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CRITICAL AND SUPERCRITICAL HIGHER ORDER PARABOLIC PROBLEMS IN $\mathbb{R}^N$

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ABSTRACT. The article is devoted to the higher order parabolic problems in $\mathbb{R}^N$ with critical or supercritical nonlinearities. For "good"-signed nonlinearities we prove the problem is globally well posed and dissipative in $L^2(\mathbb{R}^N)$ and there exists even a global attractor. On the other hand, for supercritical "bad"-signed nonlinearities we show the problem is ill posed.

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1. Introduction

In this article we consider the Cauchy problem of the form

$$\begin{cases} u_t + \Delta^2 u = f(x,u), \quad t>0, \ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x), \ x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where the nonlinear term is assumed to satisfy a certain critical or supercritical growth condition.

Critical exponents appear naturally when dealing with the well posedness of partial differential equations and they typically arise from Sobolev embeddings. These exponents describe, among others, the largest growth allowed for the nonlinear term in a given class of initial data. As such, critical exponents only account for growth of the nonlinear terms and not for its sign. Thus, they do not distinguish in general between "good" or "bad"-signed nonlinearities, e.g. for nonlinearities that for large values of u behave like $\pm|u|^{\rho-1}u$. However the sign of the nonlinear term is known to have a deep impact in the behavior of solutions of nonlinear problems.

For reaction diffusion equations

$$\begin{cases} u_t - \Delta u = f(x,u), \quad t>0, \ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x), \ x\in\mathbb{R}^N, \end{cases} \tag{1.2}$$

and considering for example initial data in the "energy" space $H^1(\mathbb{R}^N)$, if $|f(x,u)|\approx|u|^\rho$ for $|u|$ large, local existence holds for $\rho\le 1+\frac{4}{N-2}$, see e.g. [2], while if $f(x,u)\approx|u|^{\rho-1}u$ for $|u|$ large, if $\rho>1+\frac{4}{N-2}$ then (1.2) is ill posed, see [5]. Note that (1.2) has naturally associated the energy functional

$$E_{RD}(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 - \int_{\mathbb{R}^N}F(x,u),$$

where $F(x,u)=\int_0^u f(x,s)\,ds$ and if $|f(x,u)|\approx|u|^\rho$ for $|u|$ large, then typically $|F(x,u)|\approx|u|^{\rho+1}$ for $|u|$ large. Thus the critical exponent $\rho_c = 1+\frac{4}{N-2}$ in $H^1(\mathbb{R}^N)$ arises naturally as the largest value of ρ such that $H^1(\mathbb{R}^N)\subset L^{\rho+1}(\mathbb{R}^N)$. For larger ρ the nonlinear term can not be controlled by the quadratic one.

On the other hand, it is known that when $f(x,u)\approx -|u|^{\rho-1}u$ for $|u|$ large, the Cauchy problem (1.2) is well posed and dissipative for any value of ρ, see [3]. A key point in the analysis of (1.2) in [3] is that, by the maximum principle, for any value of ρ, the solution of (1.2) becomes bounded in $L^\infty(\mathbb{R}^N)$.

Observe that (1.1) has also a natural energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\Delta u|^2 - \int_{\mathbb{R}^N}F(x,u). \tag{1.3}$$

As for (1.2), the critical exponent for (1.1), $\rho_c = 1+\frac{8}{N-4}$ in $H^2(\mathbb{R}^N)$, arises naturally as the largest value of ρ such that $H^2(\mathbb{R}^N)\subset L^{\rho+1}(\mathbb{R}^N)$. For supercritical nonlinearities, if $\rho>\rho_c$ and if $f(x,u)\approx -|u|^{\rho-1}u$ for $|u|$ large, then (1.3) gives bounds on the solutions in $H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N)$. However this is not enough to prove that the solutions exists for all times, which depends strongly in proving that the solution remains in $L^\infty(\mathbb{R}^N)$, see [13].

In particular, it was proved in [13, Proposition 3.3] that if $u\in L^\infty((0,T),L^{s_0}(\mathbb{R}^N))$ and

$$s_0 > \frac{N}{4}(\rho-1) \tag{1.4}$$

then $\|u\|_{L^\infty((\varepsilon,T),L^\infty(\mathbb{R}^N))} \le K(\varepsilon, \|u\|_{L^\infty((0,T),L^{s_0}(\mathbb{R}^N))})$ for any $\varepsilon > 0$ small. But we can take $s_0 = \rho + 1$ on (1.4) only if ρ is subcritical in $H^2(\mathbb{R}^N)$.

For (1.2) the arguments in [3] and [5] mentioned above use in an essential way the maximum principle, which does not hold for fourth order equations since the kernel of the linear evolution operator changes sign, e.g. [21]. On the other hand, for the second order parabolic problems satisfying some general assumptions, by the Moser-Alikakos technique, [1], suitably weak estimate of the solutions actually implies the L^∞-bound. Neither of these, nor other techniques of getting L^∞-bound on the solutions (see e.g. [26, Theorem II.6.1]), are directly applicable to (1.1), due to the presence of the higher order terms in (1.1).

Hence, one of the motivations for this paper is to extend the results in [3] and [5] for (1.1). Problems like (1.1) drew lots of attention in the recent years; see e.g. [9, 15, 16, 17, 18, 19, 20, 12, 13] and references therein. In particular in [12] local existence of solutions for (1.1) was discussed for several classes of initial data and up to critical nonlinear terms. Also, global existence and asymptotic behavior was studied in [13] for subcritical "good"–signed nonlinear terms. On the other hand [9, 15, 16, 17, 18] payed attention to global solutions of supercritical "bad"–signed nonlinear terms. For this additional restrictions on the size and behavior at infinity of the initial data are required.

In this paper our goals are twofold. For good signed nonlinear terms, we will extend some of the results in [13] to supercritical nonlinearities. For this, we assume that the nonlinear term in (1.1) is of the general form

$$f(x,u) = g(x) + m(x)u + f_0(x,u),\ x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.5}$$

where

$$g \in L^2(\mathbb{R}^N). \tag{1.6}$$

and

$$m \in L^r_U(\mathbb{R}^N), \quad r > \frac{N}{4}, r \ge 2 \tag{1.7}$$

where the uniform space $L^r_U(\mathbb{R}^N)$, for $1 \le r \le \infty$, is defined as

$$L^r_U(\mathbb{R}^N) \overset{def}{=} \{\phi \in L^r_{loc}(\mathbb{R}^N) : \|\phi\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|\phi\|_{L^r(\{|x-y|\le 1\})} < \infty\},$$

in particular $L^\infty_U(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$ (see [4]).

For $f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ we require that

$$f_0(x,0) = 0,\ x \in \mathbb{R}^N, \tag{1.8}$$

$$|\frac{\partial f_0}{\partial u}(x,u)| \le c(1 + |u|^{\rho-1}),\ x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.9}$$

for some $c > 0$ and $\rho > 1$ such that

$$\rho \ge \frac{N+4}{N-4} = 1 + \frac{8}{N-4} \ \text{ and } \ N \ge 5. \tag{1.10}$$

and

$$\frac{\partial f_0}{\partial u}(x,u) \text{ is locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N. \tag{1.11}$$

We finally assume the structure condition

$$uf(x,u) \le C(x)u^2 + D(x)|u| - a_\rho |u|^{\rho+1}, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.12}$$

where

$$
\begin{aligned}
&C \in L^r_U(\mathbb{R}^N),\ r > \frac{N}{4},\\
&0 \le D \in L^s(\mathbb{R}^N), \quad \frac{2N}{N+4} \le s \le 2, \qquad (1.13)\\
&a_\rho \text{ is a strictly positive constant for } \rho > \frac{N+4}{N-4} \text{ and } a_\rho = 0 \text{ for } \rho = \frac{N+4}{N-4}
\end{aligned}
$$

and

$$\frac{\partial f}{\partial u}(x,u) \le L(x) \ \text{ for some } \ L \in L^\sigma_U(\mathbb{R}^N),\ \sigma > \frac{N}{4}. \qquad (1.14)$$

Note that (1.14) implies (1.12) with $a_\rho = 0$ since

$$uf(x,u) = u(f(x,u) - f(x,0)) + uf(x,0) \le L(x)u^2 + |g(x)||u| \ \text{ for } \ x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

Also the following stronger assumption implies both (1.14) and (1.12)

$$\frac{\partial f}{\partial u}(x,u) \le L(x) - a_\rho |u|^{\rho-1} \qquad (1.15)$$

for some $L \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{4}$. This applies to logistic type maps, like

$$f(x,u) = g(x) + m(x)u - n(x)|u|^{\rho-1}u,\ x \in \mathbb{R}^N, u \in \mathbb{R},$$

with $n(x) \ge n_0 > 0$ for a.e. $x \in \mathbb{R}^N$.

With these assumptions we then show that the problem (1.1) is globally well posed in $L^2(\mathbb{R}^N)$.

Theorem 1.1. *Assume (1.5)–(1.14). Then for $u_0 \in L^2(\mathbb{R}^N)$ there exists a function $u = u(\cdot; u_0)$ such that for any $T > 0$*

$$u \in C([0,T], L^2(\mathbb{R}^N)) \cap H^1((0,T), H^2(\mathbb{R}^N)) \cap L^\infty((0,T), L^{\rho+1}(\mathbb{R}^N))$$

that satisfies (1.1) in the sense that

$$\int_{\mathbb{R}^N} \big(\frac{du}{dt}v + \Delta u \Delta v - f(x,u)v\big) = 0$$

holds for a.e. $t > 0$ with any $v \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, or equivalently,

$$\frac{du}{dt} + \Delta^2 u - f(\cdot,u) = 0 \ \ in \ \ L^{\frac{\rho+1}{\rho}}_{loc}((0,\infty), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))').$$

Also, for any $\tau > 0$,

$$u(t) = e^{-\Delta^2(t-\tau)}u(\tau) + \int_\tau^t e^{-\Delta^2(t-s)} f(\cdot, u(s))ds,\ t > \tau$$

and $u(\tau) \to u_0$ in $L^2(\mathbb{R}^N)$ as $\tau \to 0$. If u_0 is sufficiently smooth, e.g. $u_0 \in C^\infty_0(\mathbb{R}^N)$, then the above holds for $\tau = 0$. Furthermore, if $u_0 \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ then

$$u : [0,T] \to H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N) \ \textit{is weakly continuous.}$$

Finally, for $u_{01}, u_{02} \in L^2(\mathbb{R}^N)$ and $T > 0$,

$$\|u(\cdot, u_{01}) - u(\cdot, u_{02})\|_{C([0,T],L^2(\mathbb{R}^N)) \cap L^2((0,T),H^2(\mathbb{R}^N))} \le c(T)\|u_{01} - u_{02}\|_{L^2(\mathbb{R}^N)},$$

for some $c(T) > 0$.

Thus

$$S(t) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad S(t)u_0 = u(t; u_0) \ \text{ for } \ t \geq 0, \ u_0 \in L^2(\mathbb{R}^N),$$

defines a C^0 semigroup $\{S(t) : t \geq 0\}$ associated to (1.1) in $L^2(\mathbb{R}^N)$.

We also establish some other properties of the semigroup $\{S(t) : t \geq 0\}$. We show that for any $t > 0$, $S(t)$ takes $L^2(\mathbb{R}^N)$ into $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ and $\{S(t) : t \geq 0\}$, restricted to $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, is a closed semigroup (thus also asymptotically closed) in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ (see [28, 11]).

If, additionally, the linear and nonlinear parts of the equation suitably cooperate; that is the solutions of the linear problem

$$\begin{cases} u_t + \Delta^2 u = C(x), \quad t > 0, \ x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.16}$$

are asymptotically decaying in $L^2(\mathbb{R}^N)$ or, equivalently,

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|^2_{L^2(\mathbb{R}^N)}, \ \phi \in H^2(\mathbb{R}^N). \tag{1.17}$$

holds for some ω_0 strictly positive (see [13, Theorem 2.1]), we show that there exists a bounded set B_0 in $L^2(\mathbb{R}^N)$ absorbing bounded sets in $L^2(\mathbb{R}^N)$ under $\{S(t) : t \geq 0\}$. Moreover, we prove

Theorem 1.2. *Assume (1.5)–(1.14) and suppose that (1.17) holds for some ω_0 strictly positive.*

Then, the semigroup associated to (1.1) in $L^2(\mathbb{R}^N)$ has a global attractor $\mathbf{A}$, that is $\mathbf{A}$ is invariant, compact in $L^2(\mathbb{R}^N)$ and

$$\sup_{b \in B} \inf_{a \in A} \|S(t)b - a\|_{H^2(\mathbb{R}^N)} \to 0 \quad as \quad t \to \infty$$

for each bounded subset B of $L^2(\mathbb{R}^N)$. In addition, $\mathbf{A}$ is bounded in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.

We also show that each individual solution somehow approaches the set of equilibria of (1.1).

On the other hand we prove that problems like (1.1) with supercritical "bad"-signed nonlinearities are generally ill posed. In particular for the problem

$$\begin{cases} u_t + \Delta^2 u = |u|^{\rho-1}u, \quad t > 0, \ x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.18}$$

whose natural energy is given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 - \frac{1}{\rho+1} \int_{\mathbb{R}^N} |u|^{\rho+1}, \tag{1.19}$$

we prove the following. Given $\rho > 1$, let u_0 be a smooth enough function. Then we say that $u(x, t)$ is a local finite energy solution of (1.18) if it is defined for some $0 \leq t < T \leq \infty$ and for each t, $u(t) \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, $u_t(t) \in L^2(\mathbb{R}^N)$, satisfies the equation in (1.18) and $t \mapsto E(u(t))$ is absolutely continuous.

Theorem 1.3. *Assume*

$$\rho > \frac{N+4}{N-4}$$

and assume u_0 is a smooth enough initial data such that

$$E(u_0) < 0$$

and there exists a local finite energy solution of (1.18).

Then (1.18) is ill posed in the sense of Hadamard in the class of finite energy solutions. More precisely there exists a sequence of smooth functions u_0^n such that

$$u_0^n \to 0 \quad in \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

with negative energy, $E(u_0^n) < 0$, and the corresponding finite energy solutions have existence times

$$T_n \to 0.$$

If

$$\rho = \frac{N+4}{N-4}$$

then (1.18) is not uniformly well posed in the class of finite energy solutions. More precisely there exists a sequence of smooth functions u_0^n such that

$$u_0^n \quad \text{is bounded in} \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

with negative energy, $E(u_0^n) < 0$, and the corresponding finite energy solutions have existence times

$$T_n \to 0.$$

We also discuss ill–possedness for other classes of solutions.

Some general comments on the assumptions above are in place. First, note that the supercritical range we consider (1.10) requires $N \geq 5$, since for $N \leq 4$ any value of ρ is subcritical. Although this may look unnatural from the physical point of view, mathematically it is interesting to understand how linear and nonlinear terms interact in equations like (1.1). Also, with assumption (1.12) with $a_\rho = 0$, global existence and asymptotic behavior of solutions was studied in [13] either for subcritical problems in $H^2(\mathbb{R}^N)$, i.e. $\rho < 1 + \frac{8}{N-4}$, or up to critical in $L^2(\mathbb{R}^N)$, i.e. $\rho \leq 1 + \frac{8}{N}$. With the stronger assumption (1.14) that paper also considered subcritical problems in $H^2(\mathbb{R}^N)$ and, as in Theorem 1.1, the solution was constructed for initial data in $L^2(\mathbb{R}^N)$. Here we have been forced to consider both (1.12) and (1.14) simultaneously, which can be obtained from (1.15). However in the particular case of critical $\rho = 1 + \frac{8}{N-4}$, our results here extend the ones in [13].

A big difference between the analysis carried out in [3] for (1.2) and the one here for (1.1) is the following. For both (1.1) and (1.2) and given some value of $\rho > 1$, it is always possible to consider smooth enough initial data such that the problem is subcritical, see e.g. [3] and [12], thus local existence of solutions follows. To prove global existence for these smooth solutions under some condition on $f(x, u)$ like (1.14), for (1.2) the maximum principle gives L^∞ bounds which do the job. However for (1.1) maximum principle does not apply and one has to rely on energy estimates. But for such smooth solutions, (1.3) does not provide enough information to prove global existence. Thus, for (1.2), one can use a density argument and extend the global solution for initial data in say, $L^2(\mathbb{R}^N)$. Therefore, here instead of just taking smoother initial data to begin with, we also regularize the equation (1.1) to make it subcritical and start the solution and prove it is global.

A brief description of the contents of this paper is as follows. In Section 2 we consider a family of approximate regularized problems and derive the estimates of approximate solutions.

In Section 3, for smooth initial data, we obtain a solution of (1.1) as the limit solution of the solutions of the regularized problems. We also prove such solution is unique in a suitable class. We then extend such solutions to initial data in $L^2(\mathbb{R}^N)$. In particular, we prove Theorem 1.1.

In Section 4 we show the solutions in Section 3 define a C^0-semigroup in $L^2(\mathbb{R}^N)$ and exhibit its dissipative properties. In particular, we complete the proof of Theorem 1.2 and that each individual solution somehow approaches the set of equilibria of (1.1), see Theorem 4.8.

Finally in Section 5 we turn our attention to the problem with "bad"-signed nonlinearity, (1.18), and prove that a local finite energy solution with negative initial energy ceases to exists in finite time, see Theorem 5.2. Using this and some scaling properties of (1.18) we prove Theorem 1.3.

Section 6 contains some final remarks concerning the supercrtitical nature of the nonlinear term.

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2. Approximate solutions and a priori estimates

In this section we consider a family of approximate regularized problems

$$\begin{cases} u_t^\eta + \eta\Lambda^{2k}u^\eta + \Delta^2 u^\eta = f(x,u^\eta), \quad t>0,\ x\in\mathbb{R}^N, \\ u^\eta(0,x) = u_0(x),\ x\in\mathbb{R}^N, \end{cases} \tag{2.1}$$

where $\eta\in(0,1]$, $4k>N$ and

$$\Lambda := -\Delta + Id. \tag{2.2}$$

We will show below that (2.1) is well posed for $u_0\in H^{2k}(\mathbb{R}^N)$ and the solutions are globally defined for $t\geq 0$. Using this, in what follows, we obtain some bounds on the solutions of (2.1). Note that we denote systematically below by c a constant that is independent of η.

We start with two preliminary results that will be used several times henceforth.

Proposition 2.1. *Suppose that $C\in L^r_U(\mathbb{R}^N)$ and $r>\max\{\frac{N}{4},1\}$.*

Then, there exists a certain $\omega_0\in\mathbb{R}$ such that (1.17) is satisfied. There exists also a continuous decreasing real valued function $\omega(\nu)$ defined in a certain interval $[0,\nu_0]$ such that

$$\lim_{\nu\to 0^+}\omega(\nu) = \omega(0) = \omega_0$$

and

$$\int_{\mathbb{R}^N}\big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) \geq \omega(\nu)\int_{\mathbb{R}^N}\phi^2 \quad \text{for all} \quad \phi\in H^2(\mathbb{R}^N),\ \nu\in[0,\nu_0].$$

Consider now the bilinear form $b: H^2(\mathbb{R}^N)\times H^2(\mathbb{R}^N)\to\mathbb{R}$

$$b(\phi,\psi) = \int_{\mathbb{R}^N} C(x)\phi\psi, \qquad \phi,\psi\in H^2(\mathbb{R}^N), \tag{2.3}$$

for which we have the following result.

Proposition 2.2. *Suppose that* $C \in L^r_U(\mathbb{R}^N)$ *with* $r \geq \frac{N}{4}$ *and* $r > 1$.
Then for some $c_0 > 0$ *we have*

$$|b(\phi, \psi)| \leq c_0 \|\phi\|_{H^2(\mathbb{R}^N)} \|\psi\|_{H^2(\mathbb{R}^N)}, \qquad \phi, \psi \in H^2(\mathbb{R}^N).$$

Proof: Suppose that $r < \infty$ as for $r = \infty$ the result is obvious.

We cover $\mathbb{R}^N$ with cubes Q_i, $i \in \mathbb{Z}^N$, centered at $i \in \mathbb{Z}^N$ and having unitary edges parallel to the axes so that $\mathbb{R}^N = \cup_{i \in \mathbb{Z}^N} \overline{Q_i}$, where $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Hence, applying a generalized Hölder's inequality with $p_1, p_2, p_3 > 1$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, we get

$$|\int_{\mathbb{R}^N} C\phi\psi| \leq \sum_{i \in \mathbb{Z}^N} \int_{Q_i} |C||\phi||\psi| \leq \sum_{i \in \mathbb{Z}^N} \|C\|_{L^{p_1}(Q_i)} \|\phi\|_{L^{p_2}(Q_i)} \|\psi\|_{L^{p_3}(Q_i)}.$$

If $N > 5$ this holds for $p_1 = \frac{N}{4}$, $p_2 = p_3 = \frac{2N}{N-4}$. If $N \leq 4$ this holds with $p_1 = r$, $p_2 = p_3 = 2r'$. In either case $L^r_U(\mathbb{R}^N) \subset L^{p_1}(Q_i)$, $H^2(Q_i) \subset L^{p_2}(Q_i) \cap L^{p_3}(Q_i)$ and we get that $|\int_{\mathbb{R}^N} C\phi\psi|$ is bounded by a multiple of $\|C\|_{L^r_U(\mathbb{R}^N)} \left(\sum_{i \in \mathbb{Z}^N} \|\phi\|^2_{H^2(Q_i)}\right)^{\frac{1}{2}} \left(\sum_{i \in \mathbb{Z}^N} \|\psi\|_{H^2(Q_i)}\right)^{\frac{1}{2}}$. Consequently, $|\int_{\mathbb{R}^N} C\phi\psi| \leq c\|C\|_{L^r_U(\mathbb{R}^N)} \|\phi\|_{H^2(\mathbb{R}^N)} \|\psi\|_{H^2(\mathbb{R}^N)}$, which proves the result. □

2.1. Estimates up to time $t = 0$.

Lemma 2.3. *Assume (1.5)–(1.13) and* $\|u_0\|_{L^2(\mathbb{R}^N)} \leq R$.
Then, for every $T > 0$, *we have*

$$\|u^\eta(t)\|^2_{L^2(\mathbb{R}^N)} \leq M_1(R, T), \qquad 0 \leq t \leq T \tag{2.4}$$

for some $M_1(R, T)$ *independent of* $\eta \in (0, 1]$.
Furthermore,

$$\|u^\eta\|^2_{L^2((0,T), H^2(\mathbb{R}^N))} + a_\rho \|u^\eta\|^{\rho+1}_{L^{\rho+1}((0,T), L^{\rho+1}(\mathbb{R}^N))} \leq M_2(R, T), \tag{2.5}$$

for some $M_2(R, T)$ *independent of* $\eta \in (0, 1]$.
If (1.17) holds for some $\omega_0 > 0$ *we have in fact*

$$\|u^\eta(t)\|^2_{L^2(\mathbb{R}^N)} \leq \|u_0\|^2_{L^2(\mathbb{R}^N)} e^{-\omega t} + M(1 - e^{-\omega t})$$

for some $M, \omega > 0$ *and independent of* η.

Proof: Multiplying (2.1) by u^η, integrating, using (1.12)-(1.13) and the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$, with s as in (1.13), we get

$$\frac{1}{2}\frac{d}{dt}\|u^\eta\|^2_{L^2(\mathbb{R}^N)} + \eta\|\Lambda^k u^\eta\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)} + a_\rho\|u^\eta\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}$$
$$\leq \int_{\mathbb{R}^N} C(x)|u^\eta|^2 + \int_{\mathbb{R}^N} D(x)|u^\eta| \leq \int_{\mathbb{R}^N} C(x)|u^\eta|^2 + \|D\|_{L^s(\mathbb{R}^N)}\|u^\eta\|_{H^2(\mathbb{R}^N)}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|u^\eta\|^2_{L^2(\mathbb{R}^N)} + \eta\|\Lambda^k u^\eta\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)} + a_\rho\|u^\eta\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}$$
$$\leq \int_{\mathbb{R}^N} C(x)|u^\eta|^2 + \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} + \frac{\varepsilon}{2}\big(\|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)} + \|u^\eta\|^2_{L^2(\mathbb{R}^N)}\big).$$

Writing $\|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)}$ as $\varepsilon\|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)} + (1-\varepsilon)\|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)}$ and using Proposition 2.1 we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^\eta\|^2_{L^2(\mathbb{R}^N)} + \eta\|\Lambda^k u^\eta\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)} + a_\rho\|u^\eta\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} + \omega\|u^\eta\|^2_{L^2(\mathbb{R}^N)} \\ \le \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)}\end{aligned} \tag{2.6}$$

for some $\omega \in \mathbb{R}$, which is positive if ω_0 in (2.3) is positive. Gronwall's lemma now leads to (2.4), while integrating (2.6) and using (2.4) we get (2.5). □

Lemma 2.4. *Besides the assumptions of Lemma 2.3 assume also (1.14) and $u_0 \in H^{4k}(\mathbb{R}^N)$. Then the solutions of* (2.1) *satisfy the estimates*

$$\|u^\eta_t(t)\|^2_{L^2(\mathbb{R}^N)} \le \|-\eta\Lambda^{2k}u_0 + (-\Delta^2+m)u_0 + f_0(\cdot,u_0) + g\|^2_{L^2(\mathbb{R}^N)}e^{-\hat{\omega}t} \tag{2.7}$$

and

$$\|u^\eta_t\|^2_{L^2((0,t),H^2(\mathbb{R}^N))} \le c\|-\eta\Lambda^{2k}u_0 + (-\Delta^2+m)u_0 + f_0(\cdot,u_0) + g\|^2_{L^2(\mathbb{R}^N)}(e^{-\hat{\omega}t}+1) \tag{2.8}$$

with certain $\hat{\omega} \in \mathbb{R}$, $c > 0$ independent of $\eta \in (0,1]$ and the initial condition u_0.

Proof: Differentiating (2.1) with respect to time we obtain that $v^\eta = u^\eta_t$ satisfies

$$\begin{cases} v^\eta_t + \eta\Lambda^{2k}v^\eta + \Delta^2 v^\eta = \frac{\partial f}{\partial u}(x,u^\eta)v^\eta, \quad t>0,\ x\in\mathbb{R}^N, \\ v^\eta(0,x) = -\eta\Lambda^{2k}u_0(x) - \Delta^2 u_0(x) + f(x,u_0(x)),\ x\in\mathbb{R}^N. \end{cases} \tag{2.9}$$

Multiplying the first equation in (2.9) by v^η, integrating and using (1.14) we get

$$\frac{1}{2}\frac{d}{dt}\|v^\eta\|^2_{L^2(\mathbb{R}^N)} + \eta\|\Lambda^k v^\eta\|^2_{L^2(\mathbb{R}^N)} + \|\Delta v^\eta\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} L(x)|v^\eta|^2 \le 0.$$

Writing $\|\Delta v^\eta\|^2_{L^2(\mathbb{R}^N)}$ as $\varepsilon\|\Delta v^\eta\|^2_{L^2(\mathbb{R}^N)} + (1-\varepsilon)\|\Delta v^\eta\|^2_{L^2(\mathbb{R}^N)}$ and using Proposition 2.1 we obtain

$$\frac{1}{2}\frac{d}{dt}\|v^\eta\|^2_{L^2(\mathbb{R}^N)} + \varepsilon\|\Delta v^\eta\|^2_{L^2(\mathbb{R}^N)} + \omega_\varepsilon\|v^\eta\|^2_{L^2(\mathbb{R}^N)} \le 0, \tag{2.10}$$

for ε small enough and some $\omega_\varepsilon \in \mathbb{R}$. This leads to (2.7) while (2.8) follows from (2.7) and (2.10). □

Now we obtain some estimates on the solutions u^η using the energy (1.3). Hence we first prove the following.

Proposition 2.5. [12, Lemma 3.1]. *If f_0 satisfies (1.8), (1.9) then there exist certain functions f_{01}, f_{02} such that*
i) $f_0(x,u) = f_{01}(x,u) + f_{02}(x,u)$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$,
ii) $f_{01}(x,0) = f_{02}(x,0) = 0$, $x \in \mathbb{R}^N$,
iii) $f_{01}(x,u)$ is globally Lipschitz in $u \in \mathbb{R}$ uniformly for $x \in R^N$ and
iv) for some $c > 0$ we have $|f_{02}(x,u_1) - f_{02}(x,u_2)| \le c|u_1-u_2|(|u_1|^{\rho-1}+|u_2|^{\rho-1})$, $u_1,u_2 \in \mathbb{R}$.
Consequently,

$$|f_{01}(x,u)| \le c|u| \quad \text{and} \quad |f_{02}(x,u)| \le c|u|^\rho \quad \text{for all} \quad u\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{2.11}$$

Using this decomposition, we have

Lemma 2.6. *If conditions (1.5)–(1.13) hold then*
i)

$$|E(\phi)| \le c(\|\phi\|_{L^2(\mathbb{R}^N)} + \|\phi\|^2_{H^2(\mathbb{R}^N)} + \|\phi\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}),\ \ \phi \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N), \tag{2.12}$$

for some constant $c > 0$.
ii) There are constants $a_1, a_2, a_3 > 0$ such that

$$a_1(\|\phi\|^2_{H^2(\mathbb{R}^N)} + a_\rho\|\phi\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}) - a_2 \le E(\phi) + a_3\|\phi\|^2_{L^2(\mathbb{R}^N)},\ \ \phi \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N). \tag{2.13}$$

Furthermore, (2.13) *holds with $a_3 = 0$ provided that (1.17) holds for some $\omega_0 > 0$.*

Proof: i) Observe that using (2.11) we get

$$|\int_{\mathbb{R}^N} F(x,\phi)| \le \int_{\mathbb{R}^N} |m(x)|\phi^2 + \int_{\mathbb{R}^N} |g(x)\phi| + \frac{c}{2}\int_{\mathbb{R}^N} |\phi(x)|^2 + \frac{c}{\rho+1}\int_{\mathbb{R}^N} |\phi(x)|^{\rho+1}.$$

Using now (1.6), Proposition 2.2 and Hölder's inequality we get (2.12).
ii) From (1.12) we now have

$$F(x,u) = \int_0^u f(x,s)\,ds \le \frac{1}{2}C(x)u^2 + D(x)|u| - \frac{a_\rho}{\rho+1}|u|^{\rho+1}, \quad u \in \mathbb{R}$$

and hence for $\phi \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$,

$$2E(\phi) \ge \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2\int_{\mathbb{R}^N} D(x)|\phi| + \frac{2a_\rho}{\rho+1}\|u\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}.$$

Using (1.13) and the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ and taking into account that the norm $\|\Delta\phi\|_{L^2(\mathbb{R}^N)} + \|\phi\|_{L^2(\mathbb{R}^N)}$ is equivalent to the $H^2(\mathbb{R}^N)$ norm we get, with Hölder's inequality,

$$\begin{aligned}
2E(\phi) - \frac{2a_\rho}{\rho+1}\|u\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} &\ge \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2\|D\|_{L^s(\mathbb{R}^N)}\|\phi\|_{L^{s'}(\mathbb{R}^N)} \\
&\ge \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} - \frac{\varepsilon}{2}(\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \|\phi\|^2_{L^2(\mathbb{R}^N)}) \\
&= \frac{\varepsilon}{2}\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \big((1-\varepsilon)|\Delta\phi|^2 - C(x)\phi^2\big) - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} - \frac{\varepsilon}{2}\|\phi\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Applying Proposition 2.1 with $\varepsilon > 0$ small enough we obtain

$$2E(\phi) - \frac{2a_\rho}{\rho+1}\|\phi\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} \ge \frac{\varepsilon}{2}\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \omega\|\phi\|^2_{L^2(\mathbb{R}^N)} - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)}$$

with some $\omega \in \mathbb{R}$, which is positive if ω_0 in (1.17) is positive. The proof now follows easily. □

Hence for the solutions of (2.1) we have

Lemma 2.7. *Assume (1.5)–(1.13) and $u_0 \in H^{4k}(\mathbb{R}^N)$. Then the solution of (2.1) satisfy, for $t \ge 0$,*

$$a_1\big(\|u^\eta(t)\|^2_{H^2(\mathbb{R}^N)} + a_\rho\|u^\eta(t)\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}\big) \le E(u_0) + \frac{\eta}{2}\|\Lambda^k u_0\|^2_{L^2(\mathbb{R}^N)} + a_2 + a_3\|u^\eta(t)\|^2_{L^2(\mathbb{R}^N)},$$

where a_1, a_2, a_3 are the constants from Lemma 2.6.

Proof: Multiplying (2.1) by u_t^η we obtain

$$\frac{d}{dt}(E(u^\eta(t)) + \frac{\eta}{2}\|\Lambda^k u^\eta(t)\|^2_{L^2(\mathbb{R}^N)}) = -\|u_t^\eta(t)\|^2_{L^2(\mathbb{R}^N)} \leq 0. \tag{2.14}$$

Integrating (2.14) we have

$$E(u^\eta(t)) + \frac{\eta}{2}\|\Lambda^k u^\eta(t)\|^2_{L^2(\mathbb{R}^N)} \leq E(u_0) + \frac{\eta}{2}\|\Lambda^k u_0\|^2_{L^2(\mathbb{R}^N)}. \tag{2.15}$$

Using now (2.13) we get the result. □

2.2. **Estimates away from zero.** Now we derive better estimates on time intervals away from $t = 0$.

Lemma 2.8. *Assume (1.5)–(1.13) and* $\|u_0\|_{L^2(\mathbb{R}^N)} \leq R$. *Then for any* $0 < \tau < T$ *the solution of* (2.1) *satisfies*

$$\|u^\eta(t)\|^2_{H^2(\mathbb{R}^N)} + a_\rho\|u^\eta(t)\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} \leq M_3(R,\tau,T) \quad \tau \leq t \leq T, \tag{2.16}$$

where $M_3(R,\tau,T)$ *does not depend on* $\eta \in (0,1]$.

If ω_0 *in (1.17) is positive then the bound above can be taken independent of* T, *i.e.* $M_3 = M_3(R,\tau)$.

Proof: Using (2.4) and, for $0 < t+\tau \leq T$, integrating (2.6) in the interval $(t, t+\tau)$, we get

$$\int_t^{t+\tau} \eta\|\Lambda^k u^\eta\|^2_{L^2(\mathbb{R}^N)},\ \int_t^{t+\tau} \|u^\eta\|^2_{H^2(\mathbb{R}^N)},\ a_\rho\int_t^{t+\tau} \|u^\eta\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} \leq c_1(R,T), \tag{2.17}$$

for some constant $c_1(R,T) > 0$ independent of $t \in [0, T-\tau]$ and $\eta \in (0,1]$. Using then (2.12) and (2.17) we have

$$\int_t^{t+\tau} (E(u^\eta) + \frac{\eta}{2}\|\Lambda^k u^\eta\|^2_{L^2(\mathbb{R}^N)}) \leq c_2(R,T) \tag{2.18}$$

with $c_2(R,T)$ independent of $\eta \in (0,1]$ and $t \in [0, T-\tau]$.

Now, for $s \in (t, t+\tau)$ integrating in (2.14) yields

$$E(u^\eta(t+\tau)) + \frac{\eta}{2}\|\Lambda^k u^\eta(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leq E(u^\eta(s)) + \frac{\eta}{2}\|\Lambda^k u^\eta(s)\|^2_{L^2(\mathbb{R}^N)} \tag{2.19}$$

and integrating (2.19) with respect to $s \in (t, t+\tau)$ and using (2.18) we conclude that

$$E(u^\eta(t+\tau)) + \frac{\eta}{2}\|\Lambda^k u^\eta(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leq \frac{c_2(R,T)}{\tau} \quad \text{for} \quad t \in [0, T-\tau]. \tag{2.20}$$

Thus, from this and (2.13) and (2.4) we have

$$a_1(\|u^\eta(t)\|^2_{H^2(\mathbb{R}^N)} + a_\rho\|u^\eta(t)\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}) \leq \frac{c_2(R,T)}{\tau} + a_2 + a_3 M_1(R,T) \quad \tau \leq t \leq T$$

and we get the result.

Observe that if ω_0 in (1.17) is positive, then the estimate in (2.4) is independent of T and so are the bounds in (2.17), (2.18) and (2.20). □

Lemma 2.9. *Assume (1.5)–(1.13) and* $\|u_0\|_{L^2(\mathbb{R}^N)} \leq R$. *Then for any* $0 < \tau < T$ *the solution of* (2.1) *satisfies*

$$\|u_t^\eta(t)\|^2_{L^2(\mathbb{R}^N)} \leq M_4(R,\tau,T), \quad \tau \leq t \leq T, \tag{2.21}$$

and

$$\|u_t^\eta\|^2_{L^2((\tau,T),H^2(\mathbb{R}^N))} \leq M_5(R,\tau,T) \tag{2.22}$$

where $M_4(R,\tau,T)$ and $M_5(R,\tau,T)$ are independent of $\eta \in (0,1]$.

Moreover, if ω_0 in (1.17) is positive, then $M_4 = M_4(R,\tau)$ and $M_5 = M_5(R,\tau)$.

Proof: We integrate in (2.14) and use (2.20) in the right hand side and (2.13) and (2.4) in the left hand side to get

$$\int_t^{t+\tau} \|u_t^\eta\|^2_{L^2(\mathbb{R}^N)} \leq c_1(R,\tau,T), \quad \tau \leq t \leq t+\tau \leq T. \tag{2.23}$$

Now, fix $s \in (t, t+\tau)$, integrate in (2.10) in $(s, t+\tau)$ neglecting the Laplacian term, and use (2.23) to obtain

$$\|u_t^\eta(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leq \|u_t^\eta(s)\|^2_{L^2(\mathbb{R}^N)} + c_2(R,\tau,T).$$

We now integrate in $s \in (t,t+\tau)$ and use (2.23) again to obtain

$$\|u_t^\eta(t+\tau)\|^2_{L^2(\mathbb{R}^N)} \leq \frac{(1+\tau)}{\tau} c_2(R,\tau,T), \quad \tau \leq t \leq T.$$

Since τ and T are arbitrary, we get (2.21). Integrating again in (2.10) and keeping the Laplacian term, we get (2.22).

Finally, if ω_0 in (1.17) is positive, the estimate in (2.20) and (2.23) are independent of T and so are (2.21) and (2.22). □

Remark 2.10. *If we assume that f does not depend on x, then L in (1.14) does not depend on x either, and then after multiplying* (2.1) *by Δu^η and integrating over $\mathbb{R}^N$ we would get a term of the form*

$$-\int_{\mathbb{R}^N} f(u^\eta)\Delta u^\eta = \int_{\mathbb{R}^N} f'(u^\eta)|\nabla u^\eta|^2 \leq L \int_{\mathbb{R}^N} |\nabla u^\eta|^2$$

and we would get from (2.1)

$$\int_{\mathbb{R}^N} |\nabla \Delta u^\eta|^2 \leq \int_{\mathbb{R}^N} u_t^\eta \Delta u^\eta - \int_{\mathbb{R}^N} f(u^\eta)\Delta u^\eta \leq \frac{1}{2}(\|u_t^\eta\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u^\eta\|^2_{L^2(\mathbb{R}^N)}) + L\|\nabla u^\eta\|^2_{L^2(\mathbb{R}^N)}.$$

Using (2.16) and (2.21) we could conclude that, given $\|u_0\|_{L^2(\mathbb{R}^N)} \leq R$ and $0 < \tau < T$,

$$\|u^\eta\|^2_{H^3(\mathbb{R}^N)} \leq M_6(R,\tau,T).$$

In particular, this gives an $L^\infty(\mathbb{R}^N)$ bound, independent of η, on the solutions of (2.1), for every $\rho > \rho_c$ in dimension $N \leq 5$.

2.3. **Existence, uniqueness and regularity of approximate solutions.** For the existence, uniqueness and regularity results for (2.1), we use [23, 29]. For this, recall that $\Lambda = -\Delta + Id$ is a self-adjoint positive definite operator in $L^2(\mathbb{R}^N)$ and so is $\Lambda^{2k} = (-\Delta+Id)^{2k}$ for every integer $k > 0$ (see [23, § 1.4, 1.6]). Thus, Λ^{2k} is a negative generator of a C^0 analytic semigroup in $X := L^2(\mathbb{R}^N)$. The fractional power spaces of this operator are given by

$$X^\alpha = D((\Lambda^{2k})^\alpha) = D(\Lambda^{2k\alpha}) = H^{4k\alpha}(\mathbb{R}^N) \text{ for each } \alpha \in (0,1), \ k \in \mathbb{N},$$

(see [10, (1.3.48), (1.3.62)]); in particular

$$X^{\frac{1}{2}} = H^{2k}(\mathbb{R}^N).$$

We now fix $k \geq 2$ large enough such that

$$2k - \frac{N}{2} > 0, \tag{2.24}$$

in which case

$$H^{2k}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N). \tag{2.25}$$

Hence, for $\eta > 0$ $A_\eta := \eta\Lambda^{2k}$, is still positive selfadjoint with the same fractional power spaces as above.

Thus, (2.1) can be written as an abstract problem

$$\begin{cases} \frac{du^\eta}{dt} + A_\eta u^\eta = \mathcal{F}(u^\eta) := -\Delta^2 u^\eta + f(\cdot, u^\eta), \quad t > 0, \\ u^\eta(0) = u_0 \end{cases} \tag{2.26}$$

and we prove the following result.

Proposition 2.11. *Assume (1.5)–(1.11) and (2.24). Then,*
i) given $u_0 \in H^{2k}(\mathbb{R}^N)$, there exists the unique global mild solution u^η of (2.1) *which satisfies $u^\eta \in C([0,\infty), H^{2k}(\mathbb{R}^N)) \cap C((0,\infty), H^{4k}(\mathbb{R}^N)) \cap C^1((0,\infty), H^{4\beta}(\mathbb{R}^N))$ for any $\beta \in [0,1)$ and u^η satisfies* (2.26)*.*
ii) If, in addition, $u_0 \in H^{4k}(\mathbb{R}^N)$ then also $u^\eta \in C([0,\infty), H^{4k}(\mathbb{R}^N))) \cap C^1([0,\infty), L^2(\mathbb{R}^N))$.
iii) Actually, if $u_0 \in H^{4k}(\mathbb{R}^N)$ then $v^\eta = \frac{du^\eta}{dt}$ satisfies

$$\begin{cases} \frac{dv^\eta}{dt} + A_\eta v^\eta = \mathcal{F}'(u^\eta)v^\eta := -\Delta^2 v^\eta + \frac{\partial f}{\partial u^\eta}(\cdot, u^\eta)v^\eta, \quad t > 0, \\ v^\eta(0) = -A_\eta u_0 + \mathcal{F}(u_0). \end{cases} \tag{2.27}$$

Proof: i) Due to (2.25) and assumptions on f, the nonlinear term $\mathcal{F}$ in (2.26) is Lipschitz continuous map on bounded sets from $H^{2k}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. Hence local existence part for (2.26) follows as in [23] (see also [10, Lemma 2.2.1 and Corollary 2.3.1]). The solution actually exists for all $t \geq 0$ because (2.15) and Lemmas 2.6, 2.3 imply that $H^{2k}(\mathbb{R}^N)$-norm of u^η does not blow up in a finite time.

For part ii) we refer the reader to [33, Theorem I.1].

iii) To obtain (2.27) observe that, by (1.11) and (2.25), $\mathcal{F}$ is of the class $C^1(H^{2k}(\mathbb{R}^N), L^2(\mathbb{R}^N))$ and its Fréchet derivative, $\mathcal{F}'(\varphi)h = -\Delta^2 h + m(\cdot)h + \frac{\partial f_0}{\partial\varphi}(\cdot,\varphi)h$, $\varphi, h \in H^{2k}(\mathbb{R}^N)$, is Lipschitz continuous on bounded sets from $H^{2k}(\mathbb{R}^N)$ into $\mathcal{L}(H^{2k}(\mathbb{R}^N), L^2(\mathbb{R}^N))$.

Therefore, $\mathcal{F}(u^\eta(\cdot))$ is $C^1((0,\infty), L^2(\mathbb{R}^N))$ and from the proof of Corollary 2.5, page 107 in [29] we get that $v^\eta = u^\eta_t$ is a mild solution of (2.27), that is

$$v^\eta(t) = e^{-A_\eta t}\big(-A_\eta u_0 + \mathcal{F}(u_0)\big) + \int_0^t e^{-A_\eta(t-s)}\mathcal{F}'(u^\eta(s))v^\eta(s)ds, \quad t > 0.$$

Since, from i), $\frac{du^\eta}{dt}(t)$ is locally Hölder continuous from $(0,\infty)$ into $H^{2k}(\mathbb{R}^N)$ (see [23, Theorem 3.5.2]), using the Lipschitz property of $\mathcal{F}'$, we obtain that $\mathcal{F}'(u^\eta(s))v^\eta(s)$ is locally Hölder continuous from $(0,\infty)$ into $L^2(\mathbb{R}^N)$. Consequently, v^η is a strong solution of (2.27), see [23, Lemma 3.5.1]. □

Remark 2.12. *Note that the regularity of u^η in Proposition 2.11 implies that all estimates in Lemmas 2.3, 2.4 2.7, 2.8 and 2.9 are valid.*

3. Limit solutions

Here we construct a unique global solution of (1.1) as a limit of solutions of (2.1) as $\eta \to 0$.

3.1. **Smooth initial conditions.** We start from the following auxiliary lemma.

Lemma 3.1. *Assume (1.5)–(1.14) and* (2.24). *Suppose also that* $u_0 \in H^{4k}(\mathbb{R}^N)$ *and* u^η *are the solutions of (2.26) through* u_0 *as in Proposition 2.11.*

Then, given a sequence of parameters $\eta \to 0^+$, *there is a subsequence of* $\{u^\eta\}$, *which for simplicity we denote the same, and there is a certain function* u *such that, for any* $T > 0$,

$$u \in L^\infty((0,T), H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)) \cap H^1((0,T), H^2(\mathbb{R}^N)), \tag{3.1}$$

and
i)

$$u^\eta \to u$$

weakly in $H^1((0,T), H^2(\mathbb{R}^N))$, *in* $L^{\rho+1}((0,T), L^{\rho+1}(\mathbb{R}^N))$ *and weak-∗ in* $L^\infty((0,T), H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N))$ *.*
ii)

$$u_t^\eta \to u_t$$

weak-∗ in $L^\infty((0,T), L^2(\mathbb{R}^N))$.

In particular $u^\eta \to u$ *in* $C([0,T], H^s_{loc}(\mathbb{R}^N))$, *for any* $s < 2$ *and* $u^\eta(t,x) \to u(t,x)$ *a.e. on* $\mathbb{R}^N$ *for each* $t \in [0,T]$ *as well as for a. e.* $(t,x) \in [0,T] \times \mathbb{R}^N$.

Proof: The weak convergence above comes from the bounds in Lemmas 2.3 and 2.4, which are uniform with respect to η. In fact the uniform bounds on the time derivatives imply the equicontinuity of u^η with values in $L^2(\mathbb{R}^N)$. This combined with the uniform bounds in $H^2(\mathbb{R}^N)$ and interpolation, gives the equicontinuity of u^η with values in $H^s(\mathbb{R}^N)$ for any $s < 2$. Finally, since the embedding $H^2(\mathbb{R}^N) \subset H^s_{loc}(\mathbb{R}^N)$ is compact, Arzelà-Ascoli arguments give the convergence in $C([0,T], H^s_{loc}(\mathbb{R}^N))$, for any $s < 2$.

Finally, using an increasing sequence of balls in $\mathbb{R}^N$ and a diagonal argument, we obtain the a.e. convergence in the statement. □

With this, we can prove the following.

Theorem 3.2. *Assume (1.5)–(1.14). Then for each* $u_0 \in H^{4k}(\mathbb{R}^N)$ *there exists a unique function such that for any* $T > 0$,

$$u = u(\cdot, u_0) \in L^\infty((0,T), H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)) \cap H^1((0,T), H^2(\mathbb{R}^N)), \quad u(0) = u_0,$$

and solving (1.1) in the sense that

$$\int_{\mathbb{R}^N} \big(\frac{du}{dt} v + \Delta u \Delta v - f(x,u)v\big) = 0 \tag{3.2}$$

holds for a.e. $t > 0$ *with any* $v \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, *or equivalently, for any* $T > 0$,

$$\frac{du}{dt} + \Delta^2 u - f(\cdot, u) = 0 \quad in \quad L^{\frac{\rho+1}{\rho}}((0,T), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))'). \tag{3.3}$$

Moreover, for each $u_{01}, u_{02} \in H^{4k}(\mathbb{R}^N)$ *and* $T > 0$ *the solutions above satisfy Lipschitz condition*

$$\|u(\cdot, u_{01}) - u(\cdot, u_{02})\|_{C([0,T], L^2(\mathbb{R}^N)) \cap L^2((0,T), H^2(\mathbb{R}^N))} \le c(T) \|u_{01} - u_{02}\|_{L^2(\mathbb{R}^N)}, \tag{3.4}$$

where $c(T)$ *is a positive constant independent of initial data.*

Proof: Note that from Proposition 2.5 and the estimates in Lemmas 2.3 and 2.4 we have that $f_{01}(\cdot,u^\eta)$ is bounded in $L^\infty((0,T),L^2(\mathbb{R}^N))$ and $f_{02}(\cdot,u^\eta)$ is bounded in $L^\infty((0,T),L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N))$ uniformly with respect to $\eta\in(0,1]$. Hence we can assume that $f_{01}(\cdot,u^\eta)$ converges weakly to F_1 in $L^2((0,T)\times\mathbb{R}^N)$ and $f_{02}(\cdot,u^\eta)$ converges weakly to F_2 in $L^{\frac{\rho+1}{\rho}}((0,T)\times\mathbb{R}^N)$.

Also, from Lemma 3.1 $f_{0j}(\cdot,u^\eta)\to f_{0j}(\cdot,u)$ a.e. on $[0,T]\times\mathbb{R}^N$ for $j=1,2$, and then Lemma 4.8 in [24], gives that $F_j=f_{0j}(\cdot,u)$ a.e. on $[0,T]\times\mathbb{R}^N$ for $j=1,2$ and $f_{01}(\cdot,u^\eta)\to f_{01}(\cdot,u)$ weakly in $L^2((0,T),L^2(\mathbb{R}^N))$ and $f_{02}(\cdot,u^\eta)\to f_{02}(\cdot,u)$ weakly in $L^{\frac{\rho+1}{\rho}}((0,T),L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N))$.

On the other hand weak continuity properties of linear continuous operators give that $\Delta^2u^\eta\to\Delta^2u$ and $m(x)u^\eta\to m(x)u$ weakly in $L^2((0,T),\big(H^2(\mathbb{R}^N)\big)')$; see e.g. page 204, [30].

So it remains to prove that $\eta\Lambda^k u^\eta\to 0$ in a weak sense. For this, take a very smooth function $\phi\in L^2((0,T),H^{4k}(\mathbb{R}^N))$ and then

$$\eta\int_0^T\int_{\mathbb{R}^N}u^\eta\Lambda^{2k}\phi\to 0$$

as $\eta\to 0$.

Therefore, passing to the limit in (2.1) we get that the limit function satisfies (3.3), which is equivalent to (3.2) by e.g. Lemma 7.4 in [30].

To prove the uniqueness, observe that if for $u_{01},u_{02}\in H^{4k}(\mathbb{R}^N)$ and $T>0$ we consider functions $u^1=u(\cdot,u_{01})$, $u^2=u(\cdot,u_{02})$ satisfying (3.1) and (3.3) then $V(t)=u^1(t)-u^2(t)$ satisfies $\frac{dV}{dt}+\Delta^2V=\big(f(x,u^1(t))-f(x,u^2(t))\big)$ in $L^{\frac{\rho+1}{\rho}}((0,T),(H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N))')$ and $V\in L^{\rho+1}((0,T),H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N))$. Hence

$$\int_{\mathbb{R}^N}\Big(\frac{dV}{dt}(t)V(t)+|\Delta V(t)|^2-(f(x,u^1(t))-f(x,u^2(t)))V(t)\Big)=0$$

for a.e. $t>0$. From (1.14) we then have, for $\varepsilon>0$,

$$\frac{1}{2}\frac{d}{dt}\|V\|^2_{L^2(\mathbb{R}^N)}+\varepsilon\|\Delta V\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\big((1-\varepsilon)|\Delta V|^2-L(x)|V|^2\big)\le 0$$

and using Proposition 2.1 we obtain, for some and $\omega_\varepsilon\in\mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}\|V\|^2_{L^2(\mathbb{R}^N)}+\varepsilon\|\Delta V\|^2_{L^2(\mathbb{R}^N)}+\omega_\varepsilon\|V\|^2_{L^2(\mathbb{R}^N)}\le 0 \tag{3.5}$$

for a.e. $t>0$. Dropping the middle term above, Gronwall's lemma gives

$$\|V(t)\|_{L^2(\mathbb{R}^N)}\le c(T)\|V(0)\|_{L^2(\mathbb{R}^N)},\quad 0\le t\le T.$$

Using this and integrating in (3.5) leads to

$$\varepsilon\int_0^T\|\Delta V\|^2_{L^2(\mathbb{R}^N)}\le c(T)\|V(0)\|^2_{L^2(\mathbb{R}^N)}$$

which proves (3.4) and, in particular, the claim on uniqueness. □

Remark 3.3. *i) Observe that the uniqueness in Theorem 3.2 implies that the whole family of approximate solutions u^η converges to u as $\eta\to 0^+$ as in Lemma 3.1.*
ii) Note that if we had that

$$u_t\in L^{\rho+1}_{loc}((0,T),L^{\rho+1}(\mathbb{R}^N))$$

then we would get, taking u_t as a test function in (3.3), the energy estimate for the limit equation

$$\frac{d}{dt}(E(u(t)) + \|u_t(t)\|^2_{L^2(\mathbb{R}^N)} = 0.$$

In fact, from Proposition 3.11 below, in the critical case we have

$$u_t \in L^{\rho+1}_{loc}((0,T), H^2(\mathbb{R}^N)),$$

and we get the energy as above; see see Proposition 3.11 below.

We also have

Corollary 3.4. *For $u_0 \in H^{4k}(\mathbb{R}^N)$, with $\|u_0\|_{L^2(\mathbb{R}^N)} \le R$ and $T > 0$, the solutions of (1.1) in Theorem 3.2 satisfy the following estimates, for $0 \le t \le T$,*
i)

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \le M_1(R,T)$$

$$\|u\|^2_{L^2((0,T),H^2(\mathbb{R}^N))} + a_\rho \|u\|^{\rho+1}_{L^{\rho+1}((0,T),L^{\rho+1}(\mathbb{R}^N))} \le M_2(R,T)$$

ii)

$$\|u_t(t)\|^2_{L^2(\mathbb{R}^N)} \le \|(-\Delta^2 + m)u_0 + f_0(\cdot,u_0) + g\|^2_{L^2(\mathbb{R}^N)} e^{-\hat{\omega}t}$$

$$\|u_t\|^2_{L^2((0,t),H^2(\mathbb{R}^N))} \le c\|(-\Delta^2 + m)u_0 + f_0(\cdot,u_0) + g\|^2_{L^2(\mathbb{R}^N)}(e^{-\hat{\omega}t} + 1),$$

for some $\hat{\omega} \in \mathbb{R}$.
iii)

$$\|u\|_{L^\infty((0,T),H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N))} \le c\big(E(u_0) + M_1(R,T) + 1\big).$$

Also, for $0 < \tau < T$, they satisfy
iv)

$$\|u\|_{L^\infty((\tau,T),H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N))} \le M_3(R,\tau,T)$$

v)

$$\|u_t\|^2_{L^\infty((\tau,T),L^2(\mathbb{R}^N))} \le M_4(R,\tau,T)$$

$$\|u_t\|^2_{L^2((\tau,T),H^2(\mathbb{R}^N))} \le M_5(R,\tau,T).$$

Additionally, if (1.17) holds for some $\omega_0 > 0$ we have

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \le \|u_0\|^2_{L^2(\mathbb{R}^N)} e^{-\omega t} + M(1 - e^{-\omega t})$$

for some $M, \omega > 0$ and in the bounds above M_1, M_3, M_4 and M_5 can be taken independent of T.

Remark 3.5. *If we have an estimate as in Remark 2.10, then weak lower semicontinuity will give again that given $\|u_0\|_{L^2(\mathbb{R}^N)} \le R$ and $0 < \tau < T$,*

$$\|u\|^2_{H^3(\mathbb{R}^N)} \le M_6(R,\tau,T).$$

which gives an $L^\infty(\mathbb{R}^N)$ bound, independent of η, for every $\rho > \rho_c$ in dimension $N \le 5$.

Proposition 3.6. *For $u_0 \in H^{4k}(\mathbb{R}^N)$ the solution of (1.1) in Theorem 3.2 satisfies the variation of constants formula*

$$u(t) = e^{-\Delta^2 t}u_0 + \int_0^t e^{-\Delta^2(t-s)} f(\cdot,u(s))ds, \; t > 0. \tag{3.6}$$

Proof: Letting $\mathcal{X} = H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, we have that $f(\cdot, u(\cdot)) \in L^{\frac{\rho+1}{\rho}}((0,T), \mathcal{X}') \hookrightarrow L^1((0,T), \mathcal{X}')$ and (3.2) implies

$$\frac{d}{dt}\langle u, v\rangle_{\mathcal{X}',\mathcal{X}} + \langle u, \Delta^2 v\rangle_{\mathcal{X}',\mathcal{X}} - \langle f(\cdot, u), v\rangle_{\mathcal{X}',\mathcal{X}} = 0$$

for a.e. $t > 0$ whenever $v \in H^4(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.

On the other hand, Δ^2 generates a C^0 analytic semigroup $\{e^{-\Delta^2 t}\}$ in $H^2(\mathbb{R}^N)$ and in $L^{\rho+1}(\mathbb{R}^N)$ (see [12]). In particular $\{e^{-\Delta^2 t}\}$ is a C^0-semigroup in $\mathcal{X} = H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ whose generator is the realization of $-\Delta^2$ in $\mathcal{X}$. Consequently, by [29, Corollary 1.10.6], $\{e^{-\Delta^2 t}\}$ is also a C^0 semigroup in $\mathcal{X}'$ (as the the adjoint semigroup generated by the adjoint infinitesimal operator).

Therefore the results in [7] gives that u satisfies (3.6). □

3.2. **Initial conditions in $L^2(\mathbb{R}^N)$.** We now obtain limit solutions for initial data in $L^2(\mathbb{R}^N)$.

Theorem 3.7. *Assume (1.5)–(1.14).*

If $u_0 \in L^2(\mathbb{R}^N)$ then for any sequence $\{u_{0n}\} \subset H^{4k}(\mathbb{R}^N)$ converging to u_0 in $L^2(\mathbb{R}^N)$, the sequence of solutions of (1.1), $\{u(t; u_{0n})\}$, as in Theorem 3.2, is a Cauchy sequence in $C([0,T], L^2(\mathbb{R}^N)) \cap L^2((0,T), H^2(\mathbb{R}^N))$ for every $T > 0$.

i) The limit function $u(\cdot; u_0)$ is independent of the sequence $\{u_{0n}\} \subset H^{4k}(\mathbb{R}^N)$ and satisfies

$$u \in C([0,T], L^2(\mathbb{R}^N)) \cap L^2((0,T), H^2(\mathbb{R}^N)),\ T > 0,$$

and for all $u_{01}, u_{02} \in L^2(\mathbb{R}^N)$ and $T > 0$,

$$\|u(\cdot, u_{01}) - u(\cdot, u_{02})\|_{C([0,T],L^2(\mathbb{R}^N))\cap L^2((0,T),H^2(\mathbb{R}^N))} \leq c(T)\|u_{01} - u_{02}\|_{L^2(\mathbb{R}^N)},$$

for some $c(T) > 0$. In particular, $u(t; u_0)$ is continuous in $L^2(\mathbb{R}^N)$ with respect to $(t, u_0) \in [0,\infty) \times L^2(\mathbb{R}^N)$.

ii) For any $T > \tau > 0$

$$u(\cdot, u_0) \in L^\infty((\tau,T), H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)) \cap H^1((\tau,T), H^2(\mathbb{R}^N))$$

and satisfies (1.1) in the sense that

$$\int_{\mathbb{R}^N} \big(\frac{du}{dt} v + \Delta u \Delta v - f(x,u)v\big) = 0$$

holds for a.e. $t > 0$ with any $v \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, or equivalently,

$$\frac{du}{dt} + \Delta^2 u - f(\cdot, u) = 0 \quad \text{in} \quad L^{\frac{\rho+1}{\rho}}_{loc}((0,\infty), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))').$$

Also, for any $\tau > 0$,

$$u(t) = e^{-\Delta^2(t-\tau)}u(\tau) + \int_\tau^t e^{-\Delta^2(t-s)} f(\cdot, u(s))ds,\ t > \tau$$

and $u(\tau) \to u_0$ in $L^2(\mathbb{R}^N)$ as $\tau \to 0$.

iii) Furthermore, if $u_0 \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ then

$$u : [0,T] \to H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N) \text{ is weakly continuous.}$$

iv) Finally, estimates in Corollary 3.4 hold as follows. Estimates i), iv) and v) hold for $u_0 \in L^2(\mathbb{R}^N)$, estimate ii) holds if $u_0 \in H^4(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ and estimate iii) holds if $u_0 \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.

Proof: Part i) follows from (3.4) while the first part of ii) follows from estimates iv) and v) in Corollary 3.4 and weak lower semicontinuity. Also, from (3.3) we have for $u^n = u(\cdot; u_0^n)$ and $0 < \tau < T$,

$$\frac{du^n}{dt} + \Delta^2 u^n - f(\cdot, u^n) = 0 \ \text{ in } \ L^{\frac{\rho+1}{\rho}}((\tau, T), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))').$$

Now observe that the estimates i), iv) and v) in Corollary 3.4 for u^n are uniform in $n \in \mathbb{N}$ and then as in the proof of Theorem 3.2 we have $u_t^n \to u_t$ weak-$*$ in $L^\infty((\tau, T), L^2(\mathbb{R}^N))$ and weakly in $L^2((\tau, T), H^2(\mathbb{R}^N))$, $f_{01}(\cdot, u^n) \to f_{01}(\cdot, u)$ weakly in $L^2((\tau, T), L^2(\mathbb{R}^N))$, $f_{02}(\cdot, u^n) \to f_{02}(\cdot, u)$ weakly in $L^{\frac{\rho+1}{\rho}}((\tau, T), L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N))$, $\Delta^2 u^n \to \Delta^2 u$ and $m(x)u^n \to m(x)u$ weakly in $L^2((0, T), \big(H^2(\mathbb{R}^N)\big)')$. Hence passing to the limit we get

$$\frac{du}{dt} + \Delta^2 u - f(\cdot, u) = 0 \ \text{ in } \ L_{loc}^{\frac{\rho+1}{\rho}}((0, \infty), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))').$$

Also, using Proposition 3.6 we get the rest.

Part iii) follows from continuity of u in $L^2(\mathbb{R}^N)$ and [31, Theorem 2.1]; see also Lemma 3.3, page 72 in [32].

Finally, part iv) follows from weak lower semicontinuity of the norms. □

Remark 3.8. *i) Note that if we had that*

$$u_t \in L_{loc}^{\rho+1}((0, T), L^{\rho+1}(\mathbb{R}^N))$$

then we would get, taking u_t as a test function in (3.3), the energy estimate for the limit equation

$$\frac{d}{dt}(E(u(t)) + \|u_t(t)\|^2_{L^2(\mathbb{R}^N)} = 0.$$

In fact, from Proposition 3.11 below, in the critical case we have

$$u_t \in L_{loc}^{\rho+1}((0, T), H^2(\mathbb{R}^N)),$$

and we get the energy as above; see Proposition 3.11 below.

Remark 3.9. *Note that for reaction diffusion equations as in (1.2), using maximum principles, one obtains that $u \in L^\infty((\tau, T), L^\infty(\mathbb{R}^N))$. Hence $f_0(\cdot, u) \in L^\infty((\tau, T), L^2(\mathbb{R}^N))$ and much further regularity of u follows by the variation of constants formula, see* [3].

Also, for (1.1), if ρ is subcritical, we also obtain that $u \in L^\infty((\tau, T), L^\infty(\mathbb{R}^N))$, see [13]. *Here we are able to obtain only that $u \in L^\infty((\tau, T), L^{\rho+1}(\mathbb{R}^N))$. Hence $f_{02}(\cdot, u) \in L^{\frac{\rho+1}{\rho}}((\tau, T), L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N))$ and the larger the ρ the worse is the estimate, since $\frac{\rho+1}{\rho} \to 1$ as $\rho \to \infty$. Also, since ρ is critical or supercritical using $u \in L^\infty((\tau, T), H^2(\mathbb{R}^N))$ and Sobolev embedding does not give better estimates either.*

To end this section we prove that in the critical case, $\rho = \frac{N+4}{N-4}$, the solution of (1.1) constructed in Theorem 3.2 for initial data in $H^{4k}(\mathbb{R}^N)$ and in Theorem 3.7, for initial data in $H^2(\mathbb{R}^N)$, coincide with the solutions constructed in [12]. This, in particular, shows that in the assumptions of the theorems above, the solutions of the critical case are globally defined. This extends the results obtained in [13].

We first recall the local existence-uniqueness result established in [12, Theorem 1.4].

Proposition 3.10. *Assume (1.5)–(1.11).*

If $\rho = \frac{N+4}{N-4}$ then (1.1) is locally well posed in $H^2(\mathbb{R}^N)$. Furthermore, the solution u of (1.1) through $u_0 \in H^2(\mathbb{R}^N)$ satisfies

$$u \in C([0,\tau_{u_0}), H^2(\mathbb{R}^N)) \cap C((0,\tau_{u_0}), H^4(\mathbb{R}^N)) \cap C^1((0,\tau_{u_0}), L^2(\mathbb{R}^N)), \tag{3.7}$$

where $[0,\tau_{u_0})$ is the maximal interval of existence of this solution.

Moreover $u \in C^1((0,\tau_{u_0}), H^{4\theta}(\mathbb{R}^N))$ for any $\theta < 1$.

Proof: The existence of solution and (3.7) come straight from [12, Theorem 1.4].

The extra C^1 regularity comes from observing that in the proof of that result one actually has that the nonlinear term is Lipschitz on bounded sets from $H^{2+\varepsilon}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ for some $\varepsilon > 0$. Hence, since $u(\tau) \in H^{2+\varepsilon}(\mathbb{R}^N)$ for any $\tau > 0$, then the smoothing effect of the equation gives the result, see e.g. Theorem 3.5.2, page 71, in [23] and [10]. □

Then we have

Proposition 3.11. *Assume (1.5)–(1.14) and suppose that $\rho = \frac{N+4}{N-4}$.*
i) For $u_0 \in H^{4k}(\mathbb{R}^N)$, with k as in (2.24), the solutions of (1.1) as in Theorem 3.2 and as in Proposition 3.10 coincide. In particular, $\tau_{u_0} = \infty$ in Proposition 3.10.
ii) For $u_0 \in H^2(\mathbb{R}^N)$, the solutions of (1.1) in Theorem 3.7 and as in Proposition 3.10 coincide. In particular, $\tau_{u_0} = \infty$ in Proposition 3.10.

In either case above, the solution of (1.1) satisfies the energy equation

$$\frac{d}{dt}(E(u(t)) + \|u_t(t)\|^2_{L^2(\mathbb{R}^N)} = 0, \quad t > 0.$$

Proof: For u_0 as in the two cases in the statement, the solution in Theorems 3.2 or 3.7 and the solution in Proposition 3.10 all satisfy (3.2) and (3.3) for $t > 0$. Hence we can proceed as in the proof of Theorem 3.2 to obtain (3.5) and from here (3.4). This implies in turn that all solution coincide.

Finally the energy estimate follows as in Remarks 3.3 and 3.8, using the regularity of the solution obtained above. □

4. Semigroup of limit solutions in $L^2(\mathbb{R}^n)$ and a global attractor

As a consequence of Theorem 3.7 we have the nonlinear semigroup $\{S(t) : t \geq 0\}$ associated with (1.1) in $L^2(\mathbb{R}^N)$; namely

$$S(t) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad S(t)u_0 = u(t; u_0) \ \text{ for } \ t \geq 0, \ u_0 \in L^2(\mathbb{R}^N), \tag{4.1}$$

where $u(t; u_0)$ is as in Theorem 3.7. In particular we have thus proved Theorem 1.1.

We also have the following.

Lemma 4.1. *Assume (1.5)–(1.14). Then,*
i) for any $t_0 > 0$ and any ball B in $L^2(\mathbb{R}^N)$ we have that $S(t_0)B$ is bounded in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$,
ii) for each $t > 0$, $S(t)$ leaves $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ invariant and $\{S(t) : t \geq 0\}$ restricted to $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ is a closed semigroup in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.

Proof: Part i) is a consequence of estimate iv) in Corollary 3.4 (see Theorem 3.7 iii)-iv)).
ii) The invariance is a consequence of part iii) in Theorem 3.7. For the closeness, if $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N) \ni u_{0n} \to u_0$ in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, $t > 0$ and $S(t)u_{0n} \to v$ in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$,

then by continuity in $L^2(\mathbb{R}^N)$, $S(t)u_{0n}$ converges in $L^2(\mathbb{R}^N)$ both to $S(t)u_0$ and to v which yields that $v = S(t)u_0$. $\square$

We also have the following.

Lemma 4.2. *For every $0 < \tau < T < \infty$, the semigroup $S(t)$ is continuous from $L^2(\mathbb{R}^N)$ into $H^2(\mathbb{R}^N)$, uniformly on $[\tau, T]$.*

Proof: Consider $u_{0n} \to u_0$ in $L^2(\mathbb{R}^N)$ and $u^n(t) = S(t)u_{0n}$, $u(t) = S(t)u_0$. From Theorem 3.7 we have that $V^n(t) = u^n(t) - u(t)$ satisfies the equation $\frac{dV^n}{dt} + \Delta^2 V^n = (f(x, u^n(t)) - f(x, u(t)))$ in $L^{\frac{\rho+1}{\rho}}((\tau, T), (H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))')$, and $V^n \in L^{\rho+1}((\tau, T), H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N))$. Thus,

$$\int_{\mathbb{R}^N} \big(\frac{dV^n}{dt}(t)V^n(t) + |\Delta V^n(t)|^2 - (f(x, u^1(t)) - f(x, u^2(t)))V^n(t)\big) = 0 \quad a.e.\ t > 0.$$

From (1.14) we then have, for $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} \big(\frac{dV^n}{dt}(t)V^n(t) + \varepsilon\|\Delta V^n(t)\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \big((1-\varepsilon)|\Delta V^n(t)|^2 - L(x)|V^n(t)|^2\big) \le 0$$

and using Proposition 2.1 we obtain, for some and $\omega_\varepsilon \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} \big(\frac{dV^n}{dt}(t)V^n(t) + \varepsilon\|\Delta V^n(t)\|^2_{L^2(\mathbb{R}^N)} + \omega_\varepsilon\|V^n(t)\|^2_{L^2(\mathbb{R}^N)} \le 0$$

for a.e. $t > 0$. Hence,

$$\varepsilon\|\Delta V^n(t)\|^2_{L^2(\mathbb{R}^N)} \le -\omega_\varepsilon\|V^n(t)\|^2_{L^2(\mathbb{R}^N)} + \|\frac{dV^n}{dt}(t)\|_{L^2(\mathbb{R}^N)}\|V^n(t)\|_{L^2(\mathbb{R}^N)} \quad a.e.\ t > 0.$$

By part iv) in Theorem 3.7 the right hand side above is bounded above, in $[\tau, T]$, by a multiple of $\|V^n(t)\|_{L^2(\mathbb{R}^N)}$ which is a continuous function of t, as is the left hand side above.

Therefore, by part i) in Theorem 3.7 we have $\|V^n(t)\|_{L^2(\mathbb{R}^N)} \to 0$, as $n \to \infty$, uniformly on $[\tau, T]$, and therefore $\|V^n(t)\|_{H^2(\mathbb{R}^N)} \to 0$, as $n \to \infty$, uniformly on $[\tau, T]$ as well. $\square$

In what follows we show that if solutions of the linear problem (1.16) are asymptotically decaying in $L^2(\mathbb{R}^N)$ the semigroup has strong dissipativeness properties. In particular, there exists a global attractor in the sense of [22].

Lemma 4.3. *Assume (1.5)–(1.14) and suppose that (1.17) holds for some $\omega_0 > 0$. Then i) there exists a ball in $L^2(\mathbb{R}^N)$ absorbing bounded sets in $L^2(\mathbb{R}^N)$ under $\{S(t) : t \ge 0\}$; that is, there exists $R_0 > 0$ such that for any bounded set $B \subset L^2(\mathbb{R}^N)$ of initial data of (1.1) and a certain $T_B \ge 0$ we have*

$$\|S(t)u_0\|_{L^2(\mathbb{R}^N)} \le R_0 \quad \text{for all } t \ge T_B \text{ and } u_0 \in B. \tag{4.2}$$

ii) there exists a bounded subset B_0 of $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ absorbing bounded subsets of $L^2(\mathbb{R}^N)$, that is

$$S(t)B \subset B_0 \quad \text{for all } t \ge t_B \text{ and } u_0 \in B$$

whenever B is bounded in $L^2(\mathbb{R}^N)$.

Thus, for each B bounded in $L^2(\mathbb{R}^N)$ the positive orbit $\gamma^+(B) = \cup_{t\ge0} S(t)B$ is eventually bounded in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.

Proof: From Theorem 3.7 the solutions satisfy Corollary 3.4 part v) and so we have

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq \|u_0\|^2_{L^2(\mathbb{R}^N)} e^{-\omega t} + c_1,$$

with $\omega > 0$, which proves (4.2).
ii) Now Lemma 4.1 and part i) gives part ii). In fact, from i), the ball of radius R_0 in $L^2(\mathbb{R}^N)$, B_{R_0}, is absorbing. Thus, $B_0 = S(1)B_{R_0}$ is bounded and absorbing in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$. □

The next result will imply the asymptotic compactness of the semigroup in $L^2(\mathbb{R}^N)$.

Lemma 4.4. *Assume (1.5)–(1.14) and suppose that (1.17) holds with some $\omega_0 > 0$.*
If B is bounded in $L^2(\mathbb{R}^N)$ then for any $\varepsilon > 0$ there exist $t_0 > 0$ and $k_0 > 0$ such that

$$\sup_{u_0 \in B} \sup_{t \geq t_0} \|u(t;u_0)\|_{L^2(\{|x|>k_0\})} < \varepsilon \tag{4.3}$$

$$\sup_{u_0 \in B} \sup_{t \geq t_0} \{\|u(t;u_0)\|_{L^{\rho+1}(\{|x|>k_0\})}, \|u(t;u_0)\|_{H^2(\{|x|>k_0\})}\} < \varepsilon \tag{4.4}$$

Consequently, from each sequence of the form $\{S(t_n)u_{0n}\}$, where $\{u_{0n}\} \subset B$ and $t_n \to \infty$, one can choose a subsequence convergent in $L^2(\mathbb{R}^N)$-norm.

Proof: We fix a smooth $\theta_0 : [0,\infty) \to [0,1]$ such that $\theta_0(z) = 0$ for $z \in [0,1]$ and $\theta_0(z) = 1$ for $z \geq 2$. We let $\theta(z) = \theta_0^4(z)$ for $z \geq 0$ and define $\phi_k(x) = \theta\big(\frac{|x|^2}{k^2}\big)$ for each $x \in \mathbb{R}^N$ and $k \in \mathbb{N}$. We also fix B bounded in $L^2(\mathbb{R}^N)$.

Using the test function $v = u\phi_k$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u^2\phi_k + \int_{\mathbb{R}^N} |\Delta u|^2\phi_k = -\int_{\mathbb{R}^N} \Delta u(u\Delta\phi_k + 2\nabla\phi_k \cdot \nabla u) + \int_{\mathbb{R}^N} uf(x,u)\phi_k. \tag{4.5}$$

Thanks to Lemma 4.3 ii) and properties of cut-off functions there exists a constant $c > 0$ such that

$$-\int_{\mathbb{R}^N} \Delta u(u\Delta\phi_k + 2\nabla\phi_k \cdot \nabla u) \leq \frac{c}{k},\ t \geq t_B.$$

Since $\Delta(u\phi_k^{\frac{1}{2}}) = \phi_k^{\frac{1}{2}}\Delta u + 2\nabla u \cdot \nabla(\phi_k^{\frac{1}{2}}) + u\Delta(\phi_k^{\frac{1}{2}})$, using Lemma 4.3 ii) we also obtain that for each $\nu > 0$ there is a certain constant, c_ν, such that

$$\int_{\mathbb{R}^N} |\Delta u|^2\phi_k = \int_{\mathbb{R}^N} \big(\Delta(u\phi_k^{\frac{1}{2}}) - 2\nabla u \cdot \nabla(\phi_k^{\frac{1}{2}}) - u\Delta(\phi_k^{\frac{1}{2}})\big)^2 \geq \int_{\mathbb{R}^N} (1 - \frac{\nu}{2})|\Delta(u\phi_k^{\frac{1}{2}})|^2 - \frac{c_\nu}{k}.$$

On the other hand from (1.12) we get

$$\int_{\mathbb{R}^N} uf(x,u)\phi_k \leq \int_{\mathbb{R}^N} C(x)(u\phi_k^{\frac{1}{2}})^2 + \int_{\mathbb{R}^N} D(x)|u|\phi_k - a_\rho \int_{\mathbb{R}^N} |u|^{\rho+1}\phi_k$$

and hence (4.5) transforms into

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u^2\phi_k + \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2\phi_k + \frac{1}{2}(1-\frac{\nu}{2})\int_{\mathbb{R}^N} |\Delta(u\phi_k^{\frac{1}{2}})|^2 + a_\rho \int_{\mathbb{R}^N} |u|^{\rho+1}\phi_k \\ \leq \int_{\mathbb{R}^N} C(x)(u\phi_k^{\frac{1}{2}})^2 + \int_{\mathbb{R}^N} D(x)|u|\phi_k + \frac{c_\nu}{2k}. \end{aligned} \tag{4.6}$$

We next have $\int_{\mathbb{R}^N} D(x)|u|\phi_k \leq \|D\phi_k^{\frac{1}{2}}\|_{L^s(\mathbb{R}^N)}\|u\phi_k^{\frac{1}{2}}\|_{L^{s'}(\mathbb{R}^N)}$ which due to (1.13) and the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ can be bounded by $\tilde{c}\|D\phi_k^{\frac{1}{2}}\|_{L^s(\mathbb{R}^N)}(\|\Delta(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)})$.

Hence we get

$$\int_{\mathbb{R}^N} D(x)|u|\phi_k \le \frac{\nu}{4}(\|\Delta(u\phi_k^{\frac{1}{2}})\|^2_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)}) + \frac{\tilde{c}^2}{\nu}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)}. \tag{4.7}$$

From (4.6)–(4.7) we have

$$\frac{1}{2}\frac{d}{dt}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N}|\Delta u|^2\phi_k + a_\rho\int_{\mathbb{R}^N}|u|^{\rho+1}\phi_k + \frac{1}{2}\int_{\mathbb{R}^N}\Big((1-\nu)|\Delta(u\phi_k^{\frac{1}{2}})|^2 - 2C(x)(u\phi_k^{\frac{1}{2}})^2\Big)$$
$$\le \frac{\nu}{4}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{\tilde{c}^2}{\nu}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)} + \frac{c_\nu}{2k}$$

and using Proposition 2.1 with $\nu > 0$ small enough so that $\omega(\nu) - \frac{\nu}{2} =: \omega > 0$ we infer that

$$\frac{1}{2}\frac{d}{dt}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N}|\Delta u|^2\phi_k + a_\rho\int_{\mathbb{R}^N}|u|^{\rho+1}\phi_k + \frac{\omega}{2}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} \le \frac{\tilde{c}^2}{\nu}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)} + \frac{c_\nu}{2k}. \tag{4.8}$$

In particular, $z_k(t) := \|u(t)\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)}$ satisfies the differential inequality $z_k'(t)+\omega z_k(t) \le c_k$, where $\omega > 0$ and $c_k = \frac{2\tilde{c}^2}{\nu}\left(\int_{\{|x|>k\}}|D|^s\right)^{\frac{2}{s}} + \frac{c_\nu}{k} \to 0$ as $k \to \infty$, which leads to (4.3).

Now in (4.8) for $t \ge t_0$ and k large enough, using the estimate in Corollary 3.4 part v) (see Theorem 3.7 iv)),

$$\Big|\frac{1}{2}\frac{d}{dt}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)}\Big| = \Big|\int_{\mathbb{R}^N} u_t u\phi_k\Big| \le \|u_t\|_{L^2(\mathbb{R}^N)}\|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)} \le C\varepsilon.$$

Thus, from (4.8), we get

$$\frac{1}{2}\int_{\mathbb{R}^N}|\Delta u|^2\phi_k + a_\rho\int_{\mathbb{R}^N}|u|^{\rho+1}\phi_k \le C\varepsilon, \quad t \ge t_0$$

and we prove the second estimate in the statement.

For $\{S(t_n)u_{0n}\}$ and $\varepsilon > 0$ we now have

$$\|S(t_n)u_{0n} - S(t_m)u_{0m}\|_{L^2(\{|x|>k_0\})} \le \varepsilon$$

for all $n, m \ge N_0$ and some $k_0, N_0 > 0$. Since, due to Lemma 4.3 ii), almost all elements of the sequence $\{S(t_n)u_{0n}\}$ lie in a bounded subset of $H^2(\mathbb{R}^N)$, then using compact embedding $H^2(\{|x| < k\}) \hookrightarrow L^2(\{|x| < k\})$ there exists a subsequence, $\{S(t_{n'})u_{0n'}\}$, which converges in $L^2(\{|x| < k\})$ for any $k \in \mathbb{N}$. Hence this subsequence is a Cauchy sequence in $L^2(\mathbb{R}^N)$ and the proof is complete. □

Now we can finally prove the following.

Theorem 4.5. *Assume (1.5)–(1.14) and suppose that (1.17) holds for some ω_0 strictly positive.*

Then, the semigroup associated to (1.1) in $L^2(\mathbb{R}^N)$ has a global attractor $\mathbf{A}$*, which is invariant, compact in $L^2(\mathbb{R}^N)$ and*

$$\sup_{b\in B}\inf_{a\in A}\|S(t)b - a\|_{H^2(\mathbb{R}^N)} \to 0 \quad as \quad t \to \infty$$

for each bounded subset B of $L^2(\mathbb{R}^N)$. In addition, $\mathbf{A}$ *is compact in $H^2(\mathbb{R}^N)$ and bounded in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$.*

Proof: From Lemmas 4.3, 4.4 we have that the semigroup is bounded dissipative and asymptotically compact in $L^2(\mathbb{R}^N)$. Thus there is a global attractor $\mathbf{A}$ in $L^2(\mathbb{R}^N)$ (see [25, 22]) which due to Lemma 4.1 i) is bounded in $H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N)$. Due to Lemma 4.2 $\mathbf{A} = S(1)\mathbf{A}$ is also compact in $H^2(\mathbb{R}^N)$.

Now we prove that the attractor $\mathbf{A}$ attracts bounded subsets of $L^2(\mathbb{R}^N)$ with respect to the Hausdorff semi-distance in $H^2(\mathbb{R}^N)$. Indeed, if this is not the case, then there is a sequence of the form $\{S(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $L^2(\mathbb{R}^N)$, $t_n \to \infty$ and $\{S(t_n)u_{0n}\}$ is separated from $\mathbf{A}$ in the topology of $H^2(\mathbb{R}^N)$. Since $\mathbf{A}$ is a global attractor in $L^2(\mathbb{R}^N)$, choosing a subsequence if necessary, we have that $\{S(t_n-1)u_{0n}\}$ converges to a certain $\psi \in \mathbf{A}$ in $L^2(\mathbb{R}^N)$. But $\mathbf{A} \subset H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ so that $\psi \in H^2(\mathbb{R}^N)$ and by continuity of $S(t)$ from $L^2(\mathbb{R}^N)$ into $H^2(\mathbb{R}^N)$ in Lemma 4.2, we conclude that $\mathbf{A} \ni S(1)\psi = \lim_{n\to\infty} S(1)S(t_n-1)u_{0n} = \lim_{n\to\infty} S(t_n)u_{0n}$ in $H^2(\mathbb{R}^N)$, which is absurd. □

Remark 4.6. *If we have a bound in $H^3(\mathbb{R}^N)$ as in Remark 3.5 and if $\rho < \frac{N+6}{N-6}$, then due to (4.4) and the compact embedding $H^3(B(r)) \hookrightarrow L^{\rho+1}(B(r))$ for balls $B(r) \subset \mathbb{R}^N$ of radius $r > 0$, the semigroup $\{S(t) : t \geq 0\}$ would be asymptotically compact in $H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N)$. This holds in particular for any $\rho > 1$ if $N \leq 6$.*

In such a case, since $\{S(t) : t \geq 0\}$ is also a closed, bounded dissipative semigroup in the latter space (see Lemmas 4.1, 4.3), there would exist then a global attractor in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, see [28, 11], *and it would coincide with the attractor $\mathbf{A}$ in Theorem 4.5. In particular, $\mathbf{A}$ in Theorem 4.5 would be then compact in $H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N)$ and would attract bounded subsets of $L^2(\mathbb{R}^N)$ with respect to the Hausdorff semidistance in $H^2(\mathbb{R}^N)\cap L^{\rho+1}(\mathbb{R}^N)$.*

Remark 4.7. *Also, from Theorem 4.5 observe that in the critical case $\rho = \rho_c = \frac{N+4}{N-4}$, ($N \geq 5$), taking initial data in $H^2(\mathbb{R}^N)$, Proposition 3.11 applies and the energy in (1.3) is a Lyapunov function for the semigroup $\{S(t) : t \geq 0\}$ in $H^2(\mathbb{R}^N)$ (see* [22, pp. 49-50]*); that is,*

(i) *$E : H^2(\mathbb{R}^N) \to \mathbb{R}$ is continuous and bounded below,*
(ii) *$E(u_0) \to \infty$ as $\|u_0\|_{H^2(\mathbb{R}^N)} \to \infty$,*
(iii) *$E(S(t)u_0)$ is nonincreasing in t for each $u_0 \in H^2(\mathbb{R}^N)$,*
(iv) *if u_0 is such that $S(t)u_0$ is defined for all $t \in \mathbb{R}$ and $E(S(t)u_0) = E(u_0)$ for $t \in \mathbb{R}$ then u_0 is an equilibrium point.*

In particular the attractor $\mathbf{A}$ in Theorem 4.5 coincides in this case with the unstable set of the set $\mathcal{E}$ of equilibria; that is $\mathbf{A} = \mathcal{W}^u(\mathcal{E})$. Also, the semigroup $S(t)$ is a gradient system, see Definition 3.8.1 in [22].

Note that if $\rho > \rho_c$ the above argument does not apply. However we will obtain below some convergence of solutions to stationary solutions (see Theorem 4.8 below).

We now show that each solution converge, forwards in time, to the set of equilibria.

Theorem 4.8. *Under the assumptions of Theorem 4.5 the set of equilibria $\mathcal{E}$ of the semigroup $\{S(t) : t \geq 0\}$ is nonempty and attracts in $H^2(\mathbb{R}^N)$ points of $L^2(\mathbb{R}^N)$. Namely, for each $u_0 \in L^2(\mathbb{R}^N)$ and any sequence $t_n \to \infty$, there is a subsequence $\{t_{n_k}\}$ and an equilibrium $\varphi \in \mathcal{E}$, such that*

$$S(t_{n_k})u_0 \to \varphi \text{ in } H^2(\mathbb{R}^N) \text{ as } k \to \infty. \tag{4.9}$$

Proof: Suppose that $u_0 \in L^2(\mathbb{R}^N)$, $t_n \to \infty$ and let $u = S(t)u_0$. Following [6, §3.5] (see also [14]) it suffices to find an auxiliary sequence $\{\tilde{t_n}\}$ such that $\tilde{t_n} \in [t_n - a, t_n - b]$ for some fixed $a, b > 0$ and $\{u(\tilde{t}_n)\}$ has a subsequence convergent in $L^2(\mathbb{R}^N)$ to equilibrium. If this is the case, letting $\tau_n := t_n - \tilde{t}_n \in [a, b]$ and using that $S(t_n)u_0 = S(\tau_n)S(\tilde{t}_n)u_0$, we can assume that $\{\tau_n\}$ has a convergent subsequence. Then using Lemma 4.2 we obtain (4.9).

From estimate v) in Corollary 3.4, we have

$$\int_\tau^\infty \|u_t\|^2_{L^2(\mathbb{R}^N)} \le M_4(R, \tau), \tag{4.10}$$

while u satisfies (1.1) as in Theorem 3.7, for all $t \in \mathbb{R}^+ \setminus \mathcal{I}_u$ and $\mathcal{I}_u$ has zero Lebesgue measure.

From (4.10), choosing a subsequence if necessary, we have that $\int_{t_n-1}^{t_n-\frac{1}{2}} \|u_t\|^2_{L^2(\mathbb{R}^N)} \to 0$. Consequently for each $n \in \mathbb{N}$ there exists a certain $\tilde{t}_n \in [t_n - 1, t_n - \frac{1}{2}]$, $\tilde{t}_n \notin \mathcal{I}_u$ such that $\|u_t(\tilde{t}_n)\|^2_{L^2(\mathbb{R}^N)} \le 4 \int_{t_n-1}^{t_n-\frac{1}{2}} \|u_t\|^2_{L^2(\mathbb{R}^N)}$ as otherwise, integrating both sides in the set $[t_n - 1, t_n - \frac{1}{2}]$ we get the contradiction. Hence,

$$\|u_t(\tilde{t}_n)\|^2_{L^2(\mathbb{R}^N)} \to 0 \ \text{ as } \ n \to \infty. \tag{4.11}$$

Hence, choosing a subsequence if necessary we can assume that for some $\varphi \in \mathbf{A} \subset H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$

$$u(\tilde{t}_n) \to \varphi \quad \text{strongly in } H^2(\mathbb{R}^N) \quad \text{ and } \quad u(\tilde{t}_n; x) \to \varphi(x) \ \text{ for a.e. } x \in \mathbb{R}^N. \tag{4.12}$$

Also, since $\{u(\tilde{t}_n)\}$ is bounded in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, see Lemma 4.3 ii), we have that $\{m(\cdot)u(\tilde{t}_n)\}$, $\{f_{01}(\cdot, u(\tilde{t}_n)\}$ and $\{f_{02}(\cdot, u(\tilde{t}_n)\}$ are bounded in $H^{-2}(\mathbb{R}^N)$, $L^2(\mathbb{R}^N)$ and in $L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N)$ respectively (see Proposition 2.2 and (2.11)). Taking again a subsequence if necessary, we can assume that

$$\begin{aligned}
&u(\tilde{t}_n) \to \chi_0 \quad \text{weakly in } H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N), \\
&f_{01}(\cdot, u(\tilde{t}_n)) \to \chi_1 \quad \text{weakly in } L^2(\mathbb{R}^N), \\
&f_{02}(\cdot, u(\tilde{t}_n)) \to \chi_2 \quad \text{weakly in } L^{\frac{\rho+1}{\rho}}(\mathbb{R}^N), \\
&\Delta u(\tilde{t}_n) \to \chi_3 \quad \text{weakly in } L^2(\mathbb{R}^N), \\
&m(\cdot)u(\tilde{t}_n) \to \chi_4 \quad \text{weakly in } H^{-2}(\mathbb{R}^N).
\end{aligned} \tag{4.13}$$

Now, using (4.12) and the continuity of f_{01}, f_{02}, and we get $f_{0j}(u(\tilde{t}_n; x)) \to f_{0j}(\varphi(x))$ a.e. $x \in \mathbb{R}^N$, for $j = 1, 2$. Thus, we actually have $\chi_0 = \varphi$, $\chi_1 = f_{01}(\cdot, \varphi)$ and $\chi_2 = f_{02}(\cdot, \varphi)$. Using weak continuity property of linear operators (see [8, Theorem III.9]) we also have $\chi_3 = \Delta\varphi$ and $\chi_4 = m(\cdot)\varphi$.

Since, we have

$$\int_{\mathbb{R}^N} \big(\frac{du}{dt}(\tilde{t}_n)v + \Delta u(\tilde{t}_n)\Delta v - gv - m(x)u(\tilde{t}_n)v - f_{01}(x, u(\tilde{t}_n))v - f_{02}(x, u(\tilde{t}_n))v\big) = 0$$

for each $n \in \mathbb{N}$ and $v \in H^2(\Omega) \cap L^{\rho+1}(\mathbb{R}^N)$, passing to the limit as $n \to \infty$ we obtain via (4.11), (4.13) that

$$\int_{\mathbb{R}^N} \big(\Delta\varphi\Delta v - gv - m(x)\varphi v - f_{01}(x, \varphi)v - f_{02}(x, \varphi)v\big) = 0 \ \text{ for any } \ v \in H^2(\Omega) \cap L^{\rho+1}(\mathbb{R}^N).$$

Hence $\varphi \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ is an equilibrium of (1.1). □

5. Finite time existence and ill posed problems

In this section we consider the Cauchy problem (1.18) with $\rho > 1$, that is

$$\begin{cases} u_t + \Delta^2 u = u|u|^{\rho-1}, & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^N, \end{cases} \tag{5.1}$$

for which the energy is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 - \frac{1}{\rho+1}\int_{\mathbb{R}^N} |u|^{\rho+1} \tag{5.2}$$

(see (1.19)).

5.1. **Finite time existence.** In what follows, using concavity method (see [27]), we prove in Theorem 5.2 that suitably smooth solutions of (5.1) corresponding to initial data with negative energy cease to exist in a finite time.

Given $\rho > 1$, let u_0 be a smooth enough function. Then we say that $u(x,t)$ is a local finite energy solution of (5.1) if it is defined for some $0 \le t < T \le \infty$ and for each t, $u(t) \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$, $u_t(t) \in L^2(\mathbb{R}^N)$, satisfies the equation in (5.1) and $t \mapsto E(u(t))$ is absolutely continuous.

Remark 5.1. *If $N \le 8$, given $\rho > 1$, the nonlinear term f in (1.1) is Lipschitz continuous map from $H^{4\alpha}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ for $\alpha < 1$ close enough to* 1*. Using then the results in* [23, Chapter 3] *(see also* [33, Theorem I.1]*) we obtain that for each $u_0 \in H^4(\mathbb{R}^N)$ there exists a function*

$$u \in C([0,\tau_{u_0}), H^4(\mathbb{R}^N)) \cap C^1([0,\tau_{u_0}), L^2(\mathbb{R}^N)) \cap C^1((0,\tau_{u_0}), H^{4\alpha}(\mathbb{R}^N))$$

satisfying in $L^2(\mathbb{R}^N)$ both relations in (5.1) as long as it exists. In particular, it is a local finite energy solution.

Theorem 5.2. *Let $\rho > 1$ and assume u_0 is a smooth enough initial data such that*

$$E(u_0) < 0.$$

Assume also that $u(x,t)$ is a local finite energy solution of (5.1).

Then u ceases to exist in a finite time.

Proof: Observe that, since the local solution is smooth enough we have,

$$\frac{d}{dt}(E(u)) = -\|u_t\|^2_{L^2(\mathbb{R}^N)},$$

which implies

$$E(u(t)) = -\int_0^t \|u_t\|^2_{L^2(\mathbb{R}^N)} + E(u_0) \quad \text{and} \quad E(u(t)) \le E(u_0).$$

On the other hand, from the equation in (5.1) we infer that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} = -\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)}$$

and using the energy (5.2) we get for $\alpha > 0$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} = \left(\frac{\alpha}{2} - 1\right)\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \left(1 - \frac{\alpha}{\rho+1}\right)\|u\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} - \alpha E(u). \tag{5.3}$$

We now choose

$$\alpha \in (2, \rho+1)$$

so that the coefficients in the parenthesis in (5.3) are positive; namely

$$\frac{\alpha}{2} - 1 > 0 \ \text{ and } \ 1 - \frac{\alpha}{\rho+1} > 0$$

in which case

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} \geq -\alpha E(u(t)) = \alpha \int_0^t \|u_t\|^2_{L^2(\mathbb{R}^N)} - \alpha E(u_0) \geq -\alpha E(u_0). \tag{5.4}$$

In particular, if $E(u_0) < 0$ then

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} \geq 2\alpha \int_0^t \|u_t\|^2_{L^2(\mathbb{R}^N)} > 0 \tag{5.5}$$

and (5.4) yields

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \geq -2\alpha E(u_0)t + \|u_0\|^2_{L^2(\mathbb{R}^N)} =: R(t) \tag{5.6}$$

and the right hand side $R(t) \to \infty$ as $t \to \infty$.

Now we define

$$M(t) = \int_0^t \|u\|^2_{L^2(\mathbb{R}^N)}$$

and observe that (5.5) reads

$$\frac{d^2 M}{dt^2}(t) \geq 2\alpha \int_0^t \|u_t\|^2_{L^2(\mathbb{R}^N)}.$$

Also, we have

$$M(t)\frac{d^2 M}{dt^2}(t) \geq 2\alpha \left(\int_0^t \int_{\mathbb{R}^N} u u_t\right)^2 = \frac{\alpha}{2}\left(\int_0^t \frac{d}{dt}(\|u\|^2_{L^2(\mathbb{R}^N)})\right)^2 = \frac{\alpha}{2}\left(\frac{dM}{dt}(t) - \frac{dM}{dt}(0)\right)^2$$

and hence

$$M(t)\frac{d^2 M}{dt^2}(t) > (\frac{\alpha}{2} - \varepsilon)\left(\frac{dM}{dt}(t)\right)^2, \tag{5.7}$$

provided that

$$\varepsilon \in (0, \frac{\alpha}{2}) \quad \text{ and } \quad \frac{dM}{dt}(t) > \frac{\alpha}{2\varepsilon}\left(1 + \sqrt{1 + \frac{2\varepsilon}{\alpha}}\right)\frac{dM}{dt}(0). \tag{5.8}$$

Assume the local finite energy solution, u, exists for all $t > 0$. Then (5.6) implies that $M(t) \to \infty$ and $\frac{dM}{dt}(t) \to \infty$ as $t \to \infty$. In particular (5.8) holds for $t \geq t_0$. But now (5.7) implies that $0 \leq M^\beta(t)$ is concave with $\beta = 1 + \varepsilon - \frac{\alpha}{2} < 0$, provided $0 < \varepsilon < \frac{\alpha}{2} - 1$. Since $M^\beta(t) \to 0$ as $t \to \infty$, this is a contradiction. □

Remark 5.3. *Note that, since $\rho > 1$, for any nontrivial $u_0 \in H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ examining the energy along the ray of u_0,*

$$E(su_0) = \frac{|s|^2}{2}\int_{\mathbb{R}^N} |\Delta u_0|^2 - \frac{|s|^{\rho+1}}{\rho+1}\int_{\mathbb{R}^N} |u_0|^{\rho+1}$$

we have that $E(0) = 0$, while $E(su_0) > 0$ for $s \in (0, s_0)$ and $E(su_0) < 0$ for $s > s_0$, where s_0 depends on u_0.

5.2. **Ill posed supercritical problems.** In this section we give evidence that if ρ is supercritical then (5.1) is ill posed.

For this we start with the following simple lemma.

Lemma 5.4. *Assume $\rho > 1$ and $u(x,t)$ is a smooth solution of (5.1) for $x \in \mathbb{R}^N$ and $0 < t < T$.*

Then for $\lambda > 0$ and

$$\alpha = \frac{4}{\rho - 1},$$

the rescaled function

$$u_\lambda(t,x) = \lambda^\alpha u(\lambda^4 t, \lambda x), \qquad x \in \mathbb{R}^N, \quad 0 < t < \frac{T}{\lambda^4} \tag{5.9}$$

is also a solution of (5.1).

Proof: Observe that u_λ satisfies

$$(u_\lambda)_t + \Delta^2 u_\lambda = \lambda^{\alpha+4}\big(u_t + \Delta^2 u\big) = \lambda^{\alpha+4-\alpha\rho}|u_\lambda|^{\rho-1}u_\lambda$$

and the rest is immediate since $\alpha + 4 - \alpha\rho = 0$. □

Then we have the following ill–possedness result.

Theorem 5.5. *Assume*

$$\rho > \frac{N+4}{N-4}$$

and assume u_0 is a smooth enough initial data such that

$$E(u_0) < 0$$

and there exists a local finite energy solution of (5.1).

Then (5.1) is ill posed in the sense of Hadamard in the class of finite energy solutions. More precisely there exists a sequence of smooth functions u_0^n such that

$$u_0^n \to 0 \quad \text{in} \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

with negative energy, $E(u_0^n) < 0$, and the corresponding finite energy solutions have existence times

$$T_n \to 0.$$

If

$$\rho = \frac{N+4}{N-4}$$

then (5.1) is not uniformly well posed in the class of finite energy solutions. More precisely there exists a sequence of smooth functions u_0^n such that

$$u_0^n \quad \text{is bounded in} \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

with negative energy, $E(u_0^n) < 0$, and the corresponding finite energy solutions have existence times

$$T_n \to 0.$$

Proof: By Theorem 5.2 the local finite energy solution is only defined for $0 < t < T < \infty$.

Consider then the rescaled solution $u_\lambda(x,t)$ as in Lemma 5.4, which has initial data

$$u_\lambda(0) = \lambda^\alpha u_0(\lambda x), \quad \alpha = \frac{4}{\rho - 1}.$$

Now, as $\lambda \to \infty$, the solution u_λ is defined for $0 < t < \frac{T}{\lambda^4} \to 0$, while the initial data satisfies

$$\|u_\lambda(0)\|_{L^2(\mathbb{R}^N)} = \lambda^{\alpha - \frac{N}{2}} \|u_0\|_{L^2(\mathbb{R}^N)},$$

$$\|\Delta u_\lambda(0)\|_{L^2(\mathbb{R}^N)} = \lambda^{2+\alpha-\frac{N}{2}} \|\Delta u_0\|_{L^2(\mathbb{R}^N)},$$

$$\|u_\lambda(0)\|_{L^{\rho+1}(\mathbb{R}^N)} = \lambda^{\alpha - \frac{N}{\rho+1}} \|u_0\|_{L^{\rho+1}(\mathbb{R}^N)}$$

and

$$E(u_\lambda(0)) = \frac{\lambda^{2(\alpha+2-\frac{N}{2})}}{2} \int_{\mathbb{R}^N} |\Delta u_0|^2 - \frac{\lambda^{\alpha(\rho+1)-N}}{\rho+1} \int_{\mathbb{R}^N} |u_0|^{\rho+1} = \lambda^{\alpha(\rho+1)-N} E(u_0) < 0,$$

because

$$2(\alpha + 2 - \frac{N}{2}) = \alpha(\rho+1) - N.$$

Now when $\rho > \frac{N+4}{N-4}$ then

$$2 + \alpha - \frac{N}{2} < 0, \quad \alpha - \frac{N}{\rho+1} < 0 \quad \text{and} \quad \alpha(\rho+1) - N < 0$$

hence

$$u_\lambda(0) \to 0 \quad \text{in} \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

and we get the result.

On the other hand, if $\rho = \frac{N+4}{N-4}$

$$2 + \alpha - \frac{N}{2} = 0, \quad \text{and} \quad \alpha(\rho+1) - N = 0$$

hence

$$u_\lambda(0) \quad \text{is bounded in} \quad H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$$

and we get the result. □

Concerning some other classes of solutions, we have the following. Following [12] the critical exponent in $L^q(\mathbb{R}^N)$, for $1 < q < \infty$, is

$$\rho_c = 1 + \frac{4q}{N},$$

while the critical exponent in $H^2_q(\mathbb{R}^N)$ is

$$\rho_c = 1 + \frac{4q}{N - 2q}.$$

That means that if $\rho \leq \rho_c$ and $u_0 \in X$, where $X = L^q(\mathbb{R}^N)$ or $H^2_q(\mathbb{R}^N)$ then there exists a suitable local solution of (5.1).

So, given $\rho > 1$ and $u_0 \in X$, we say $u(x,t)$ is a local X solution of (5.1) if it is defined for some $0 \leq t < T \leq \infty$, $u(t) \in X$ and satisfies the equation in (5.1).

Remark 5.6. *For example, if $m \in L^\infty(\mathbb{R}^N)$, $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $N \leq 7$ and $u_0 \in H^4(\mathbb{R}^N)$, the solution mentioned in Remark 5.1 is an $L^q(\mathbb{R}^N)$ solution for $2 \leq q < \infty$.*

If moreover $2 \leq q \leq \frac{2N}{N-4}$ then it is an $H^2_q(\mathbb{R}^N)$ solution.

Theorem 5.7. *Let X be either $L^q(\mathbb{R}^N)$ or $H^2_q(\mathbb{R}^N)$.*

Assume that $\rho > \rho_c(X)$ and there exists $u_0 \in X$ and a local X solution that ceases to exist in a finite time T.

Then (5.1) is ill posed in the sense of Hadamard in the class of X solutions. More precisely there exists a sequence of smooth functions u_0^n such that

$$u_0^n \to 0 \quad in \quad X$$

and the corresponding local X solutions have existence times

$$T_n \to 0.$$

If $\rho = \rho_c(X)$ there exists a sequence of smooth functions u_0^n such that

$$u_0^n \quad is\ bounded\ in \quad X$$

and the corresponding local X solutions have existence times

$$T_n \to 0.$$

Proof: With the solution $u(x,t)$ in the statement, consider then the rescaled solution $u_\lambda(x,t)$ as in Lemma 5.4, which has initial data

$$u_\lambda(0) = \lambda^\alpha u_0(\lambda x), \quad \alpha = \frac{4}{\rho - 1}.$$

Now, as $\lambda \to \infty$, the solution u_λ is defined for $0 < t < \frac{T}{\lambda^4} \to 0$, while the initial data satisfies

$$\|u_\lambda(0)\|_{L^q(\mathbb{R}^N)} = \lambda^{\alpha - \frac{N}{q}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

Assume first $X = L^q(\mathbb{R}^N)$. Then, when $\rho > \rho_c$ we have

$$\alpha - \frac{N}{q} < 0$$

and then

$$\|u_\lambda(0)\|_{L^q(\mathbb{R}^N)} = \lambda^{\alpha - \frac{N}{q}} \|u_0\|_{L^q(\mathbb{R}^N)} \to 0$$

and we get the result.

When $\rho = \rho_c$ we have $\alpha - \frac{N}{q} = 0$ and then

$$\|u_\lambda(0)\|_{L^q(\mathbb{R}^N)} = \|u_0\|_{L^q(\mathbb{R}^N)}$$

is bounded and we get the result.

Assume now that $X = H^2_q(\mathbb{R}^N)$. We have now that

$$\|u_\lambda(0)\|_{L^q(\mathbb{R}^N)} = \lambda^{\alpha - \frac{N}{q}} \|u_0\|_{L^q(\mathbb{R}^N)},$$

$$\|\Delta u_\lambda(0)\|_{L^q(\mathbb{R}^N)} = \lambda^{2 + \alpha - \frac{N}{q}} \|\Delta u_0\|_{L^q(\mathbb{R}^N)}.$$

Now when $\rho > \rho_c$

$$\alpha + 2 - \frac{N}{q} < 0$$

and then

$$\|u_\lambda(0)\|_{H^2_q(\mathbb{R}^N)} \leq \lambda^{\alpha + 2 - \frac{N}{q}} \|u_0\|_{H^2_q(\mathbb{R}^N)} \to 0 \quad \text{as } \lambda \to \infty.$$

When $\rho = \rho_c$ we have $\alpha + 2 - \frac{N}{q} = 0$ and then

$$\|u_\lambda(0)\|_{H^2_q(\mathbb{R}^N)} \leq \|u_0\|_{H^2_q(\mathbb{R}^N)}$$

is bounded and we get the result. □

Remark 5.8. *Note that similar arguments have been used for second order problems in* [5]*. In such a case more complete results have been given due again to the maximum principle.*

Remark 5.9. *It remains an open problem if there exists an $L^q(\mathbb{R}^N)$-solution or an $H^2_q(\mathbb{R}^N)$ solution that actually ceases to exist in finite time.*

From Proposition 3.2 in [13] *we know that if for a solution of (5.1), we have a bound in finite time in $L^{s_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $s_0 > 1$, then we obtain bounds in $H^4_s(\mathbb{R}^N)$ for all $s \geq s_0$. In particular, the solution exists up to that time. Also from Gronwall's type argument, a bound in finite time in $L^\infty(\mathbb{R}^N)$ implies a bound in $L^q(\mathbb{R}^N)$, if $u_0 \in L^q(\mathbb{R}^N)$, and hence the solution exists up to that time.*

Therefore, if an $L^q(\mathbb{R}^N)$-solution (or an $H^2_q(\mathbb{R}^N)$ solution) ceases to exist in finite time then for all $s > 1$ the norm in $L^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ becomes unbounded in finite time.

6. Final remarks

As stressed in the Introduction a main difference between the solutions of the second order problem (1.1) and the second order one (1.2) is that for supercritical good signed nonlinear terms the solutions of the latter become bounded in space, due to comparison arguments.

As a consequence that for solutions of (1.1) we can not find such bound, we have not been able to obtain results one would obtain for (1.2).

For example, in Theorems 3.2 and 3.7 no further smoothing than $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$ was obtained for the solutions. Also, except for the critical case, see Proposition 3.11, we could not derive in general the energy of the solutions, see Remark 3.8.

On the other had, concerning the asymptotic behavior of solutions, we could not find asymptotic compactness in $L^{\rho+1}(\mathbb{R}^N)$ despite the tail estimates in Lemma 4.4. Only $H^2(\mathbb{R}^N)$ asymptotic compactness was achieved. In particular we could not prove the attractor in Theorem 4.5 attracts in $L^{\rho+1}(\mathbb{R}^N)$. Analogously in Theorem 4.8 the convergence to equilibria could not be proved in $L^{\rho+1}(\mathbb{R}^N)$ either.

All these stem from the fact that, in the supercrticial regime, we can not control the nonlinear term with the linear diffusion one.

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